The Wave Equation

UBC M257/316 Lecture Notes ©2014 by Philip D. Loewen

Many PDE's describe wave motion. The simplest is

$$u_{tt} = c^2 u_{xx}.$$

Here $c > 0$ is a given constant, and $u = u(x,t)$ is some quantity of interest at position x and time t. Depending on the application of interest, u could represent sound pressure in air (or water, or rock), microwave amplitude in a waveguide, electrical potential (voltage) in a cable, or lateral string displacement in a musical instrument.

Units. Recall that $[z]$ means "the units of z". Suppose $[t] = \mathrm{s}$, $[x] = \mathrm{m}$. Then

$$\left[\frac{\partial^2 u}{\partial t^2}\right] = \left[c^2 \frac{\partial^2 u}{\partial x^2}\right] \iff \frac{[u]}{\mathrm{s}^2} = [c]^2 \frac{[u]}{\mathrm{m}^2} \iff [c]^2 = \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2.$$

No matter what units u has, c has units of **velocity.** Does this mean something?

A. Derivation—Motion of an Elastic String

Consider an elastic string whose equilibrium position is on the x-axis. Linear density of the string is ρ; typically $[\rho] = \mathrm{kg/m}$. The string can move, but only transversally, i.e., each particle moves at right angles to x-axis. Write $u(x,t)$ for the displacement at time t of the particle whose equilibrium position is x.

Bowed String on $[0, \ell]$

Let $\tau = \tau(x,t)$ denote tension in string at position x, time t: note $\tau(x,t) \geq 0$ for all x, t. ("You can't push on a rope.")

Let $\theta = \theta(x,t)$ denote the angle of elevation for the tangent line at point x: note $-\pi/2 < \theta(x,t) < \pi/2$ for each x, and $\tan\theta(x,t) = u_x(x,t)$.

Assumptions. (i) Transverse motion only (see above).

(ii) Small slopes: for all $x \in (0, \ell)$, slope $u_x(x,t)$ is so small that

$$\cos\theta(x,t) \approx 1, \qquad \sin\theta(x,t) \approx \frac{\sin\theta(x,t)}{\cos\theta(x,t)} = \tan\theta(x,t) = u_x(x,t).$$

[Note: $\tan\theta - \sin\theta = \left[\theta + \frac{\theta^3}{3} + \dots\right] - \left[\theta - \frac{\theta^3}{6} \pm \cdots\right] = \frac{\theta^3}{2} + O(\theta^5)$ extremely small for $\theta \approx 0$.]

Transverse Force Balance (u-direction). Study forces on string segment based in interval $[x, x+h]$ at instant t. Assuming $\theta > 0$ here, we have ...
Right end $(x+h)$: Tension force pulls *right* and *up*. Vertical component has magnitude

$$\tau(x+h,t)\sin\theta(x+h,t) \approx \tau(x+h,t)u_x(x+h,t) \qquad \text{(by (ii))}.$$

Left end (x): Tension force pulls *left* and *down*. Vertical component has magnitude

$$\tau(x,t)\sin\theta(x,t) \approx \tau(x,t)u_x(x,t) \qquad \text{(by (ii))}.$$

Interior points: External force per unit length in u-direction is F_{ext}, some function of u, x, t.
Newton II: $ma = F_{\text{total}}$, so

$$\rho h u_{tt}(x_{\text{CM}}, t) \approx \tau(x+h,t)u_x(x+h,t) - \tau(x,t)u_x(x,t) + hF_{\text{ext}}. \qquad (*)$$

Taking $h \to 0^+$ gives $0 = 0$ (wow). Dividing by $h > 0$ and then sending $h \to 0^+$ gives

$$\lim_{h\to 0^+} \left[\rho u_{tt}(x_{\text{CM}}, t)\right] = \lim_{h\to 0^+} \left[\frac{\tau(x+h,t)u_x(x+h,t) - \tau(x,t)u_x(x,t)}{h} + F_{\text{ext}}\right].$$

This leads to the **general wave equation**

$$\rho(x)u_{tt}(x,t) = \frac{\partial}{\partial x}\left(\tau(x,t)u_x(x,t)\right) + F_{\text{ext}}.$$

[Variable linear density is OK.]

Longitudinal Force Balance (x-direction). Net force in x-direction is

$$F = \tau(x+h,t)\cos\theta(x+h,t) - \tau(x,t)\cos\theta(x,t) + hG_{\text{ext}},$$

where G_{ext} is some external force per unit length, possibly depending on u, x, t. To sustain assumption (i) [transverse motion only], this force must be 0:

$$\tau(x+h,t)\cos\theta(x+h,t) - \tau(x,t)\cos\theta(x,t) = -hG_{\text{ext}}.$$

Simplest Case. If $G_{\text{ext}} = 0$ (no external forces), divide by h and send $h \to 0$ to get

$$0 = \frac{\partial}{\partial x}\tau(x,t)\cos\theta(x,t),$$

whence $\tau(x,t)\cos\theta(x,t) = f(t)$ must be independent of x. We are assuming $\cos\theta \approx 1$, so $\tau(x,t) = f(t)$ is independent of x. Without externally changing the tension, its variation will be due exclusively to stretching of the string, which we are neglecting. Thus $\tau(x,t)$ will actually be a constant.

When tension $\tau > 0$ and density $\rho > 0$ are both constant, we get

$$u_{tt} = \frac{\tau}{\rho}u_{xx}.$$

This fits pattern above, with $c^2 = \tau/\rho$. Check the units:

$$\left[\frac{\tau}{\rho}\right] = \frac{\text{N}}{\text{kg/m}} = \frac{\text{kg m/s}^2}{\text{kg/m}} = \left(\frac{\text{m}}{\text{s}}\right)^2.$$

As expected, $c = \sqrt{\tau/\rho}$ has units of velocity: Good.

Interesting Alternative. If the x-axis points straight up, then the external force acting on $[x, x+h]$ is its weight: this gives

$$hG_{\text{ext}} = -mg = -\rho hg.$$

In this case we would have

$$\tau(x+h,t)\cos\theta(x+h,t) - \tau(x,t)\cos\theta(x,t) = \rho hg.$$

Dividing by $h > 0$ and sending $h \to 0^+$ gives

$$\frac{\partial}{\partial x}\left(\tau(x,t)\cos\theta(x,t)\right) = \rho g.$$

Hence $\tau(x,t)\cos\theta(x,t) = \rho g x + C$ for some C. Again using $\cos\theta \approx 1$ and assuming τ has no time-dependence, we get

$$\tau(x) = \tau(0) + \rho g x.$$

Finally, if the string is hanging freely with its loose end at level $x = 0$, then $\tau(0) = 0$, so $\tau(x) = \rho g x$. In this case our wave equation (assuming constant density ρ) becomes

$$\rho u_{tt} = \frac{\partial}{\partial x}\left(\rho g x u_x\right), \qquad \text{i.e.,} \qquad u_{tt} = g\frac{\partial}{\partial x}\left(x u_x\right) = g u_x + g x u_{xx}.$$

I have seen this outside my office window! We'll study this in detail later, if time permits.

B. Boundary Conditions and Boundary Value Problems

Boundary Conditions. If string segment $[0, h]$ has left end attached to a spring with force constant k (Hooke's Law), arguments above give this modification of $(*)$:

$$\rho h u_{tt}(x_{\text{CM}}, t) \approx \tau(h,t)u_x(h,t) - ku(0,t) + hF_{\text{ext}}.$$

Taking $h \to 0^+$ here gives $0 = \tau(0,t)u_x(0,t) - ku(0,t)$; in constant-tension case one gets

$$\tau u_x(0,t) - ku(0,t) = 0.$$

Three cases arise:

(i) $k = 0$ [Free vertical motion at $x = 0$]: $u_x(0,t) = 0$.

(ii) $k \to \infty$ [Clamped string (no motion) at $x = 0$]: $u(0,t) = 0$.

(iii) $0 < k < \infty$ [Some springy resistance at $x = 0$]: $u_x(0,t) - \alpha u(0,t) = 0$. ($\alpha = k/\tau$.)

Other Interpretations. For pipes of air, $u(x,t)$ measures the excess pressure above ambient at location x and time t. Therefore $u(0,t) = 0$ if the end of the pipe at $x = 0$ is open (e.g., a flute has both ends open), whereas $u_x(0,t) = 0$ if the end of the pipe at $x = 0$ is closed (e.g., blowing across the open top of a soda bottle). Reference: Iain G. Main, Vibrations and Waves in Physics (3/e), pages 181–182.

Boundary Value Problem. A well-formed problem for the 1D wave equation in $0 < x < \ell$ has 3 ingredients. Thinking about the lateral vibrations of a tight string makes each one seem reasonable.

(PDE) – expresses Newton's 2nd law: concavity drives acceleration.

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \ell,\ t > 0.$$

[More general forms possible here.]

(BC) – force-balance relation at each end of string. Choose between

(i) $u(0,t) = 0 = u(\ell,t)$ ... fixed ends,

(ii) $u_x(0,t) = 0 = u_x(\ell,t)$ ... free ends,

(iii) $u_x(0,t) - au(0,t) = 0 = u_x(\ell,t) - bu(\ell,t)$ ... springy ends,

(iv) $u(0,t) = \phi(t)$, $u(\ell,t) = \psi(t)$ ... end positions decreed by prescribed fcns of time [e.g., skipping rope]

(v) $\tau u_x(0,t) = \phi(t)$, $\tau u_x(\ell,t) = \psi(t)$ ... given time-varying vertical forces act on ends of rope.

Cdx (i)–(iii) are homogeneous; (iv)–(v) are not [unless $\phi = 0 = \psi$]. Allow different cdx at different ends.

(IC) – initial conditions. Need two:

(i) $u(x,0) = f(x)$, $0 < x < \ell$ ... initial position,

(ii) $u_t(x,0) = g(x)$, $0 < x < \ell$ ... initial velocity.

Both f and g must be given in problem statement.

C. Separation of Variables and Modes

Consider the simplest wave equation for $0 < x < \ell$, ignoring initial conditions for now.

(PDE)	$u_{tt} = c^2 u_{xx}$,	$0 < x < \ell, \quad t > 0$,
(BC)	$x = 0$: Choose fixed ($u = 0$) or free ($u_x = 0$),	$t > 0$,
	$x = \ell$: Choose fixed ($u = 0$) or free ($u_x = 0$),	$t > 0$,

Trying $u(x,t) = X(x)T(t)$ in (PDE) leads to

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = 444 \tan\left(\frac{\pi\alpha}{2}\right), \tag{$*$}$$

for some $\alpha \in (-1,1)$. This form of the separation constant is deliberately chosen to be ugly, in order to make a point. (But notice that any real constant can be expressed this way, for some $\alpha \in (-1,1)$.) Split $(*)$:

$$T''(t) - 444c^2 \tan\left(\frac{\pi\alpha}{2}\right) T(t) = 0, \tag{1}$$

$$X''(x) - 444 \tan\left(\frac{\pi\alpha}{2}\right) X(x) = 0. \tag{2}$$

Different (BC) above have different consequences for X. E.g., suppose both ends fixed: $u(0,t)=0=u(\ell,t)$ gives

$$X(0)T(t)=0, \quad X(\ell)T(t)=0, \qquad t>0.$$

This forces $X(0)=0=X(\ell)$, so we have an eigenvalue problem for X:

$$X''(x)-444\tan\left(\frac{\pi\alpha}{2}\right)X(x)=0,\ 0<x<\ell; \qquad X(0)=0=X(\ell).$$

The eigenfunctions are known (FSS): $X_n(x)=\sin\left(\frac{n\pi x}{\ell}\right)$ for $n=1,2,3,\ldots$. The corresponding values of α are too horrible to contemplate. But the eigenfunctions are all we need to know to express

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{\ell}\right)$$

for some T_n's. Plug this into PDE:

$$\begin{aligned}
0 &= u_{tt}-c^2u_{xx}\\
&= \sum_{n=1}^{\infty}T_n''(t)\sin\left(\frac{n\pi x}{\ell}\right)-c^2\sum_{n=1}^{\infty}T_n(t)\left[-\left(\frac{n\pi}{\ell}\right)^2\right]\sin\left(\frac{n\pi x}{\ell}\right)\\
&= \sum_{n=1}^{\infty}\left[T_n''(t)+\left(\frac{n\pi c}{\ell}\right)^2T_n(t)\right]\sin\left(\frac{n\pi x}{\ell}\right).
\end{aligned}$$

By FSS theory, this gives (for each n)

$$0=\left[T_n''(t)+\left(\frac{n\pi c}{\ell}\right)^2T_n(t)\right] \implies T_n(t)=A_n\cos\left(\frac{n\pi c}{\ell}t+\delta_n\right)$$

for some amplitude constant A_n and phase shift constant δ_n determined by the initial conditions. So a typical solution [basic wave eq'n, fixed-end case] will look like

$$u(x,t)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi c}{\ell}t+\delta_n\right)\sin\left(\frac{n\pi x}{\ell}\right).$$

Notice: We never had to know the horrible α_n's in $(*)$ and use them in (1). The analogue of (1) came up naturally when plugging our series form in the PDE, and the appropriate constants were provided automatically.

Modes. Simplest nontrivial solutions have exactly one nonzero term of product form, like

$$\cos\left(\frac{5\pi c}{\ell}t\right)\sin\left(\frac{5\pi x}{\ell}\right) \qquad \text{[choose } \delta_5=0,\ A_5=1, \text{ all other } A_n=0],$$

$$\cos\left(\frac{15\pi c}{\ell}t\right)\sin\left(\frac{15\pi x}{\ell}\right) \qquad \text{[choose } \delta_5=-\pi/2,\ A_{15}=1, \text{ all other } A_n=0].$$

For fixed n, the basic eigenfunction shape $X_n(x) = \sin\left(\dfrac{n\pi x}{\ell}\right)$ is preserved but with a time-varying amplitude that varies harmonically. The time-variation (what you **hear**) has angular frequency of $\dfrac{n\pi c}{\ell}$; the mode shape variation (what you **see**) has angular frequency $\dfrac{n\pi}{\ell}$. Note that when discussing oscillatory signals in time, like sound waves or radio waves, we usually don't use angular frequency (radians per second); instead we use Hertz (cycles per second). Since one cycle equals 2π radians, the time-variations associated with motion in mode n have frequency

$$\frac{n\pi c}{\ell}\,\frac{\text{rad}}{\text{sec}} = \frac{n\pi c}{\ell}\,\frac{\text{rad}}{\text{sec}} \times \frac{1 \text{ cycle}}{2\pi \text{ rad}} = \frac{nc}{2\ell}\,\frac{\text{cycle}}{\text{sec}} = \frac{nc}{2\ell}\text{ Hz.}$$

D. Fourier's Ring and Travelling Waves

In classic wave-motion problems driven by

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \ell,\ t > 0,$$

we are well acquainted with boundary conditions that will lead to eigenfunction series from one of the "Big Four" families. Separating $u(x,t) = X(x)T(t)$ works out as follows:

$$\text{[FSS]}\ \ u(0,t) = 0 = u(\ell,t) \implies X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),\ n = 1,2,3,\ldots.$$

$$\text{[FCS]}\ \ u(0,t) = 0 = u(\ell,t) \implies X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right),\ n = 0,1,2,3,\ldots.$$

$$\text{[HPSS]}\ \ u(0,t) = 0 = u_x(\ell,t) \implies X_n(x) = \sin\left(\frac{(2n-1)}{2}\frac{\pi x}{\ell}\right),\ n = 1,2,3,\ldots.$$

$$\text{[HPCS]}\ \ u(0,t) = 0 = u_x(\ell,t) \implies X_n(x) = \cos\left(\frac{(2n-1)}{2}\frac{\pi x}{\ell}\right),\ n = 1,2,3,\ldots.$$

In problems with FSS or FCS eigenfunctions, the choice $\ell = \pi$ is convenient because then the eigenfunction families reduce to $\sin(nx)$ and $\cos(nx)$. For HPSS and HPCS problems, the convenient choice is $\ell = \pi/2$, because then the eigenfunction families become $\sin((2n-1)x)$ and $\cos((2n-1)x)$. With the right value of ℓ, the appropriate postulate for each of these problems becomes some special case of the following general pattern:

$$u(x,t) = \tfrac{1}{2}A_0(t) + \sum_{n=1}^{\infty}\left[A_n(t)\cos(nx) + B_n(t)\sin(nx)\right]. \qquad (*)$$

In detail, choosing all $A_n(t) = 0$ gives the FSS postulate when $\ell = \pi$; choosing all $B_n(t) = 0$ gives the FCS postulate for $\ell = \pi$; choosing all $A_n(t) = 0$ and all $B_k(t) = 0$ when k is even gives the HPSS postulate for $\ell = \pi/2$; choosing all $B_n(t) = 0$ and all $A_k(t) = 0$ when k is even gives the HPCS postulate for $\ell = \pi/2$. But $(*)$ has another, more direct, interpretation: it's also the natural FFS postulate for the problem on $-\ell < x < \ell$ with periodic boundary conditions. So that setup, informally known as "Fourier's Ring", captures all the others.

Infinite Extent. At each fixed instant t, the function $u(x,t)$ in line $(*)$ is certain to be 2π-periodic with respect to x. (This is true for any choice of time-varying coefficients $A_n(t)$ and $B_n(t)$.) Therefore $u(x,t)$ is actually defined for all real x, not just the x-values in the physical interval of interest. And, as we will soon show, the behaviour of solutions in the physical segment gets easier to predict once we understanding the behaviour of $u(x,t)$ on the whole real line.

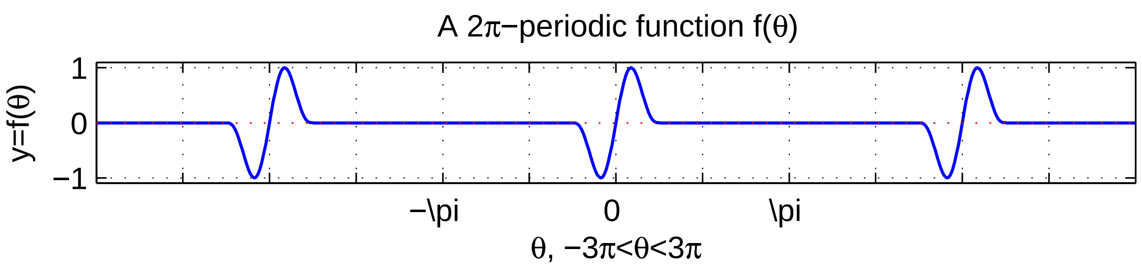
Periodicity and Visualization. For any 2π-periodic function $\widetilde{f}$, the identity

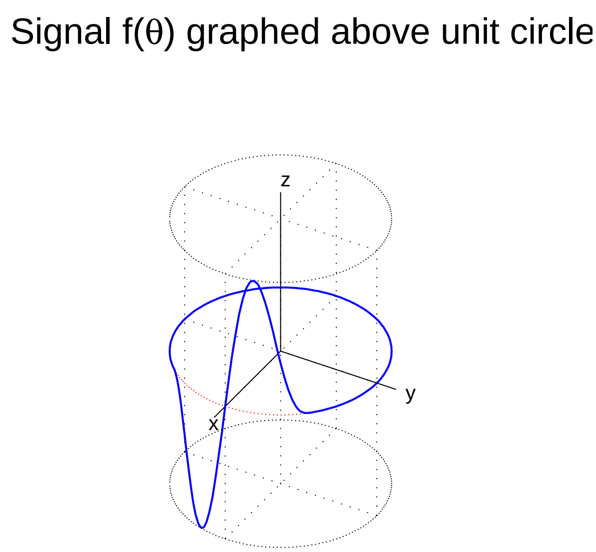
$$\widetilde{f}(x+2\pi) = \widetilde{f}(x), \qquad x \in \mathbb{R}, \tag{0}$$

means that knowing the values of $\widetilde{f}$ on any interval of length 2π is equivalent to knowing the values of $\widetilde{f}$ everywhere. To illustrate this, it's convenient to change the letter x to θ, and to imagine that θ is the central angle that labels a particular point on the unit circle. Instead of stretching out a representative interval like $-\pi < \theta \leq \pi$ onto a horizontal axis, we imagine wrapping it around the circle and joining the ends. Now (assuming $\widetilde{f}$ is continuous) the graph of $\widetilde{f}$ has the parametric representation

$$(\cos\theta, \sin\theta, \widetilde{f}(\theta)), \qquad \theta \in \mathbb{R}.$$

On the graph, the periodicity identity labelled (0) above (where now $x = \theta$) corresponds to the fact that the angles θ and $\theta + 2\pi$ select the same point on the unit circle. Here are two views of the same function to illustrate the idea:





With these observations in mind, we now change the letter x to θ and imagine $u = u(\theta, t)$ having the unit circle for its domain. This exactly captures the periodicity property of u in the FFS series form ($*$). As noted above, each of the Big Four eigenfunction series solutions can then be obtained by insisting on 0-values for certain collections of coefficient functions, and by restricting the physical interval of interest to a suitable sub-arc of the whole circle. The physical domains for each case are shown here:

[FFS] the whole ring, $-\pi < \theta < \pi$;

[FSS] half the ring, $0 < \theta < \pi$ (recall $\ell = \pi$);

[FCS] half the ring, $0 < \theta < \pi$ (recall $\ell = \pi$);

[HPSS] the quarter-ring $0 < \theta < \pi/2$ (recall $\ell = \pi/2$);

[HPCS] the quarter-ring $0 < \theta < \pi/2$ (recall $\ell = \pi/2$).

Modes. In ($*$), the "modes" are the functions $\cos(nx)$ and $\sin(nx)$. Plugging ($*$) into the wave equation shows how each mode moves:

$$\ddot{A}_n(t) + n^2c^2A_n(t) = 0, \qquad \ddot{B}_n(t) + n^2c^2B_n(t) = 0.$$

For problems with *zero initial velocity*, both ODE families have cosine-type solutions, and we have

$$u(\theta, t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nct)\cos(n\theta) + b_n \cos(nct)\sin(n\theta)\right]. \qquad (*)$$

This is an infinite superposition of simple separated pieces, and in each one the index n influences both the mode shape and the frequency with which it oscillates in time.

Travelling Waves. Recall that in line ($*$), $u_t(\theta, 0) = 0$ for all θ. Define

$$\widetilde{f}(\theta) = u(\theta, 0) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

Then recall the identities

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B), \qquad (\dagger)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B). \qquad (\ddagger)$$

Adding the two equations condensed into the single line in ($\dagger$) gives

$$\boxed{\sin(A)\cos(B) = \tfrac{1}{2}\left[\sin(A - B) + \sin(A + B)\right].}$$

For the two equations packed into line ($\ddagger$), adding and subtracting both produce useful results:

$$\boxed{\begin{aligned} \sin(A)\sin(B) &= \tfrac{1}{2}[\cos(A - B) - \cos(A + B)], \\ \cos(A)\cos(B) &= \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]. \end{aligned}}$$

Adapt these for use in $(*)$: with $A = n\theta$ and $B = nct$, we have

$$\cos(nct)\cos(n\theta) = \tfrac{1}{2}\left[\cos(n[\theta - ct]) + \cos(n[\theta + ct])\right],$$
$$\cos(nct)\sin(n\theta) = \tfrac{1}{2}\left[\sin(n[\theta - ct]) + \sin(n[\theta + ct])\right].$$

Splitting $a_0 = \frac{1}{2}a_0 + \frac{1}{2}a_0$ then gives

$$\begin{aligned} u(\theta, t) = \tfrac{1}{2}&\left(\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n[\theta - ct]) + b_n\sin(n[\theta - ct])\right]\right) \\ &\tfrac{1}{2}\left(\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n[\theta + ct]) + b_n\sin(n[\theta + ct])\right]\right) \\ &= \tfrac{1}{2}\widetilde{f}(\theta - ct) + \tfrac{1}{2}\widetilde{f}(\theta + ct). \end{aligned}$$

Shifts. For any number p, the curve $y = g(x - p)$ is an exact copy of $y = g(x)$, shifted to the right by p units. Thus

$$y = g(x - ct)$$

is an exact copy of $y = g(x)$, shifted right by $p = ct$ units. That is, the shift increases with time at the constant rate c: the whole curve moves to the right, with speed c, and no change in shape. Likewise $y = g(x + ct)$ represents a curve moving to the left, with speed c, no change in shape.

In our wave-equation problem, the profile $\widetilde{f}(\theta) = u(\theta, 0)$ splits into two equal parts $\widetilde{f} = \frac{1}{2}\widetilde{f} + \frac{1}{2}\widetilde{f}$, and then these parts move to the left and the right at speed c. The movies in class clearly show this!

Exercise. Show that for general interval $0 < x < \ell$, and for each Big-Four eigenfunction family, a wave BVP with zero initial velocity is solved by the appropriate version of

$$u(x, t) = \tfrac{1}{2}\widetilde{f}(x - ct) + \tfrac{1}{2}\widetilde{f}(x + ct), \qquad \text{where} \quad f(x) = u(x, 0),\ 0 < x < \ell.$$

Unwinding. After understanding how $u(\theta, t)$ moves in graphs based on the unit circle, it is time to undo the change of variables $x = \theta$ and think again about stretching the variable x out along the straight infinite real axis. Now $u(x, 0)$ admits a 2π-periodic extension called $\widetilde{f}(x)$, defined for all real x, and this function defines the shapes that slide sideways along the real line.

Graphical Solutions. To solve a standard string problem with any combination of fixed and free ends, *and zero initial velocity, ...*

(1) Build the extension $\widetilde{f}$ appropriate to the eigenfunctions of the problem. This provides the initial condition for the whole real axis.

(2) Half the extended initial condition $\widetilde{f}$ will travel to the right with speed c, the other half to the left; neither shape will change. At each instant t, the sum of these two contributions gives the solution for u. This "mathematical motion"

happens on the entire x-axis, and the segment in the interval $0 < x < \ell$ correctly captures the actual "physical motion" on the interval of interest. In particular, the BC's will always be satisfied.

(3) Reflections [good exam question]: Pulses hitting a pinned end bounce back upright; pulses hitting a free end bounce back inverted. (Optics applications: Newton's Rings, 3D glasses in mirror.)

Verification. For $u(x,t) = \frac{1}{2}\widetilde{f}(x+ct) + \frac{1}{2}\widetilde{f}(x-ct)$, we have $u(x,0) = \widetilde{f}(x) = f(x)$ at every point of the basic interval, while the Chain Rule gives

$$u_t(x,t) = \frac{c}{2}\widetilde{f}'(x+ct) - \frac{c}{2}\tfrac{1}{2}\widetilde{f}'(x-ct) \implies u_t(x,0) = \frac{c}{2}\widetilde{f}'(x) - \frac{c}{2}\tfrac{1}{2}\widetilde{f}'(x) = 0.$$

Thus the initial conditions check out. The Chain rule also shows that this function satisfies the PDE. Only the BC's present a real challenge. We'll show two cases here, and encourage readers to practice on the others.

Fixed/Fixed. In problems where the BC's are $u(0,t) = 0 = u(\ell,t)$, the FSS is appropriate. So in the solution form

$$u(x,t) = \tfrac{1}{2}\widetilde{f}(x+ct) + \tfrac{1}{2}\widetilde{f}(x-ct),$$

the function $\widetilde{f}$ is odd and 2ℓ-periodic. Consequently

$$\begin{aligned} u(0,t) &= \tfrac{1}{2}\widetilde{f}(ct) + \tfrac{1}{2}\widetilde{f}(-ct) = 0 \qquad \text{(since } \widetilde{f} \text{ is odd)},\\ u(\ell,t) &= \tfrac{1}{2}\widetilde{f}(\ell+ct) + \tfrac{1}{2}\widetilde{f}(\ell-ct) = \tfrac{1}{2}\widetilde{f}(\ell+ct) + \tfrac{1}{2}\widetilde{f}(-\ell-ct) = 0\\ &\qquad\qquad \text{(since } \widetilde{f} \text{ is } 2\ell\text{-periodic and odd)}. \end{aligned}$$

Free/Free. In problems where the BC's are $u_x(0,t) = 0 = u_x(\ell,t)$, the FCS is appropriate. So in the solution form

$$u(x,t) = \tfrac{1}{2}\widetilde{f}(x+ct) + \tfrac{1}{2}\widetilde{f}(x-ct),$$

the function $\widetilde{f}$ is even and 2ℓ-periodic. This implies that $\widetilde{f}'$ is odd and 2ℓ-periodic. The Chain Rule gives $u_x(x,t) = \frac{1}{2}\widetilde{f}'(x+ct) + \frac{1}{2}\widetilde{f}'(x-ct)$. Consequently

$$\begin{aligned} u_x(0,t) &= \tfrac{1}{2}\widetilde{f}'(ct) + \tfrac{1}{2}\widetilde{f}'(-ct) = 0 \qquad \text{(since } \widetilde{f}' \text{ is odd)},\\ u_x(\ell,t) &= \tfrac{1}{2}\widetilde{f}'(\ell+ct) + \tfrac{1}{2}\widetilde{f}'(\ell-ct) = \tfrac{1}{2}\widetilde{f}'(\ell+ct) + \tfrac{1}{2}\widetilde{f}'(-\ell-ct) = 0\\ &\qquad\qquad \text{(since } \widetilde{f}' \text{ is odd)}. \end{aligned}$$

E. D'Alembert's Solution

For any two smooth functions ϕ and ψ of one variable, possibly unrelated to each other, the function

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \qquad (*)$$

will satisfy the wave equation $u_{tt} = c^2 u_{xx}$. This is easy to check using the Chain Rule. But there's more: *Every smooth solution of the wave equation has the form* $(*)$

for well-chosen ϕ and ψ. We have already seen how the choices $\phi = \psi = \frac{1}{2}\widetilde{f}$ come up in separation-of-variables solutions associated with all of our favourite eigenfunction families. Now we will show that the more general form in $(*)$ covers every conceivable solution.

Suppose $u = u(x,t)$ is a solution of the wave equation ... smooth enough that $u_{xt} = u_{tx}$ everywhere. Introduce new variables

$$r = x - ct, \qquad s = x + ct;$$

observe $x = \frac{1}{2}(s+r)$, $t = \dfrac{1}{2c}(s-r)$. Let

$$U(r,s) = u(x,t) = u\left(\frac{s+r}{2}, \frac{s-r}{2c}\right).$$

Then use the Chain Rule to calculate

$$U_r(r,s) = u_x\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[\frac{1}{2}\right] + u_t\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[-\frac{1}{2c}\right]$$

and similarly

$$\begin{aligned} U_{rs}(r,s) = \left[\frac{1}{2}\right]&\left\{u_{xx}\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[\frac{1}{2}\right] + u_{xt}\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[\frac{1}{2c}\right]\right\} \\ &+ \left[-\frac{1}{2c}\right]\left\{u_{tx}\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[\frac{1}{2}\right] + u_{tt}\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)\left[\frac{1}{2c}\right]\right\} \\ = \frac{1}{4}u_{xx}&\left(\frac{s+r}{2}, \frac{s-r}{2c}\right) - \frac{1}{4c^2}u_{tt}\left(\frac{s+r}{2}, \frac{s-r}{2c}\right) = 0. \end{aligned}$$

This shows $\dfrac{\partial}{\partial s}U_r = 0$, so U_r is independent of s, i.e., $U_r(r,s) = \Phi(r)$. This forces $U(r,s) = \phi(r) + \psi(s)$ for some smooth ϕ and ψ. Conclusion: Every smooth solution of $u_{tt} = c^2 u_{xx}$ can be expressed as

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \qquad (**)$$

for suitable choices of ϕ and ψ. This expression has a legitimate claim to be called "the general solution" of $u_{tt} = c^2 u_{xx}$.

Matching Given IC's. Now suppose IC's $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are given. What ϕ and ψ should be used in $(**)$? First,

$$f(x) = u(x,0) = \phi(x) + \psi(x) \qquad \text{for all } x. \qquad (1)$$

Similarly,

$$g(x) = u_t(x,0) = \left[-c\phi'(x-ct) + c\psi'(x+ct)\right]_{t=0}, \quad \text{so} \quad \psi'(x) - \phi'(x) = \frac{1}{c}g(x).$$

This is required to hold for all real x. So we write it again with the dummy variable p replacing x and then integrate:

$$\begin{aligned}\int_{p=0}^{x} \left[\psi'(p) - \phi'(p)\right]\, dp &= \frac{1}{c}\int_{p=0}^{x} g(p)\, dp \\ \iff \psi(x) - \phi(x) &= \psi(0) - \phi(0) + \frac{1}{c}\int_{p=0}^{x} g(p)\, dp.\end{aligned} \tag{2}$$

Now add equations (1) and (2) to get

$$\psi(x) = \tfrac{1}{2} f(x) + \tfrac{1}{2}\left[\psi(0) - \phi(0)\right] + \frac{1}{2c}\int_{0}^{x} g(p)\, dp.$$

Subtract (it's similar) and the result is

$$\phi(x) = \tfrac{1}{2} f(x) - \tfrac{1}{2}\left[\psi(0) - \phi(0)\right] - \frac{1}{2c}\int_{0}^{x} g(p)\, dp.$$

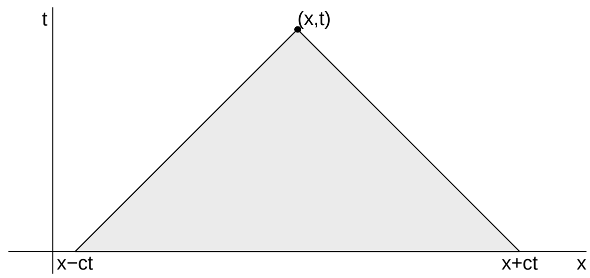
Once again, these formulas are valid for all real x, so we can replace x throughout by any real-valued expression. Thus when we use these forms in the solution form $(**)$, we get

$$\begin{aligned}u(x,t) &= \phi(x - ct) + \psi(x + ct) \\ &= \tfrac{1}{2} f(x - ct) + \tfrac{1}{2} f(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\, dr.\end{aligned}$$

This is **D'Alembert's Solution** for $u_{tt} = c^2 u_{xx}$ on the whole x-axis with initial displacement f and initial velocity g.

Reconciliation. How does D'Alembert's solution relate to the problem on a finite string? Before answering, we must note that the f in D'Alembert's solution must have all of $\mathbb{R}$ for its domain in order to use that formula for large t-values. Let's consider the case $g = 0$. Then to start from the eigenfunction series and and express that solution in form $(**)$ requires doing an appropriate eigenfunction extension and applying D'Alembert's formula to the function $\widetilde{f}$, as outlined above. To work backwards, supposing that a solution of form $(**)$ obeys the given BC's and respects the condition of zero initial velocity, one can deduce that u must have the form written above, and $\widetilde{f}$ must have the expected periodicity and symmetry properties. (Supplementary homework and practice problems.)

Domain of Influence. An observer at the point P with position x listening at time t will observe an amplitude $u(x,t)$ determined by the the initial displacement f at the points ct units to the left and right of x, together with a contribution combining initial velocity information between the two. This can be illustrated by drawing a two-dimensional cone: information in the shaded region influences the value $u(x,t)$, while information outside it gets ignored. (As time advances, the cone moves upward, and includes more and more of the (x,t) plane. So information that was previously ignored eventually reaches the observer. Information travels at speed c.)


t
(x,t)
x−ct
x+ct
x