II. Series Solutions for Ordinary Differential Equations

A. Series Solutions around Ordinary Points

Generic Example. Find two power series solutions around $x = 0$ for

$$y'' + xy' + y = 0.$$

Solution. Write $y = \sum_k a_k x^k$, $y' = \sum_k k a_k x^{k-1}$, $y'' = \sum_k k(k-1) a_k x^{k-2}$. Tabulate terms in the given ODE and usd substitution to identify the coefficient of x^n:

Term	Series	Sub	x^n-coeff
y''	$\sum_k k(k-1)a_k x^{k-2}$	$k-2=n$ $k=n+2$	$(n+2)(n+1)a_{n+2}$
xy'	$\sum_k k a_k x^k$	$k=n$	na_n
y	$\sum_k a_k x^k$	$k=n$	a_n

Adding these expressions will give another power series:

$$y'' + xy + y = \sum_n c_n x^n,$$

where $\quad c_n = (n+2)(n+1)a_{n+2} + na_n + a_n = (n+1)\left[(n+2)a_{n+2} + a_n\right].$

To get $y'' + xy + y = 0$, the identity theorem requires $c_n = 0$ for each $n \geq 0$, i.e.,

$$a_{n+2} = -\frac{1}{n+2} a_n, \qquad n \geq 0.$$

This **recurrence relation** lets us find all the coefficients. Initialize $a_0 = 0$, $a_1 = 1$ and calculate; then repeat starting from $a_0 = 1$, $a_1 = 0$. Solutions:

$$a_0 = 1,\ a_1 = 0 \quad \Longrightarrow \quad y_1 = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} - \frac{x^{10}}{3840} \pm \cdots$$

$$a_0 = 0,\ a_1 = 1 \quad \Longrightarrow \quad y_2 = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - \frac{x^{11}}{10395} \pm \cdots$$

Both have $\rho = +\infty$, by theory described later. It's clear that y_1 and y_2 are linearly independent, so the general solution of the given equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \qquad c_1, c_2 \in \mathbb{R}.$$

Notice: y_1 is an even function, y_2 is an odd function. ////

Theorem. *If functions p and q are analytic at x_0, then the differential equation*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

has two linearly independent solutions y_1 and y_2. Each has the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

and each has a radius of convergence that satisfies

$$\rho(y; x_0) \geq \min \left\{ \rho(p; x_0),\ \rho(q; x_0) \right\}.$$

Calculation. To find the two solutions mentioned in the Theorem,

(i) Confirm that the theorem applies, by comparing the given equation to prototype (1). To do this, transform the equation into standard form and then verify that the resulting coefficient functions p and q really are analytic at x_0.

(ii) Reorganize the given equation to clear fractions if possible. Then expand all coefficients in power series centred at x_0.

(iii) Postulate a series-form solution in the form promised by the theorem. Write, explicitly,

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

Plug this series form for y into (1), and use algebraic methods to reshape the left side of (1) into a power series:

$$y''(x) + p(x)y'(x) + q(x)y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n. \tag{2}$$

Expect each coefficient c_n to be a linear combination of the (unknown) coefficients a_k in the postulated form for y. (*Error Control*: It is imperative that each c_n be independent of x.)

(iv) Together, equations (1) and (2) require $c_n = 0$ for each n. Use this to derive a recurrence relation between the coefficients a_k. (*Error Control*: A correctly-formulated recurrence relation will not contain the independent variable x.)

(v) Choose $a_0 = 1$ and $a_1 = 0$ and use the recurrence relation to produce specific values for $a_2, a_3, \ldots$, in the first solution

$$y_1(x) = 1 + 0(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots.$$

Then choose $a_0 = 0$ and $a_1 = 1$ and use the recurrence relation again to produce new values for $a_2, a_3, \ldots$, in the second solution

$$y_2(x) = 0 + (x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots.$$

Note that $y_1(x_0) = 1$, $y_1'(x_0) = 0$, while $y_2(x_0) = 0$, $y_2'(x_0) = 1$.

Indexing Observation. The derivative formula

$$\frac{d}{dx}\left[\sum_k a_k(x-x_0)^k\right] = \sum_k k a_k (x-x_0)^{k-1}$$

holds whenever $x \neq x_0$ for summation over *any finite collection of integers k* (allowing $k > 0$, $k = 0$, and $k < 0$). It remains valid for sums over *all integers k* in regions where the corresponding series converge. [General series of this form are called Laurent series. They are discussed in detail in courses on complex analysis like UBC Math 300.] Working with Laurent Series instead of Taylor Series sounds more general and more difficult, but in fact it saves work for us. We can safely write sums over all integers k, and then simply remember that for a power series,

$$\boxed{a_k = 0 \text{ for all } k < 0.}$$

This way we don't have to pay special attention to the initial indices in the power series. (Many textbooks waste a lot of effort on this.)

Example. Find two linearly independent solutions valid near $x_0 = 1$:

$$xy'' + y' + xy = 0. \tag{†}$$

Solution. Dividing by x produces the standard form

$$y'' + \left(\frac{1}{x}\right)y' + y = 0, \tag{‡}$$

in which we see that the coefficients have just one complex singularity, namely, $x = 0$. This lies a distance 1 from the given expansion centre $x_0 = 1$, so we can be certain that power series solutions of the desired form exist and have radius of convergence at least 1. (In symbols, $\rho \geq 1$.)

We seek solutions in the form $y = \sum_k a_k(x-1)^k$, with $a_k = 0$ whenever $k < 0$. Substitution into the standard form (‡) is nasty: the fraction-free form (†) given in the question is easier to handle. But even here, some further processing is required. Transform the coefficients of y'' and of y using

$$x = 1 + (x-1).$$

This is step (ii) in the recipe given above, where we replace the coefficient functions with their power series in the quantity $(x-1)$. This produces a five-term equation in which substitution works cleanly:

$$0 = y + (x-1)y + y' + y'' + (x-1)y''.$$

Plugging the postulated solution into the ODE gives the terms tabulated below.

Term	Series	Sub	$(x-1)^n$-coeff
y	$\sum_k a_k(x-1)^k$	$k=n$	a_n
$(x-1)y$	$\sum_k a_k(x-1)^{k+1}$	$k+1=n$ $k=n-1$	a_{n-1}
y'	$\sum_k ka_k(x-1)^{k-1}$	$k-1=n$ $k=n+1$	$(n+1)a_{n+1}$
y''	$\sum_k k(k-1)a_k(x-1)^{k-2}$	$k-2=n$ $k=n+2$	$(n+2)(n+1)a_{n+2}$
$(x-1)y''$	$\sum_k k(k-1)a_k x^{k-1}$	$k-1=n$ $k=n+1$	$(n+1)na_{n+1}$

Adding these expressions will give a single power series for the derivative combination in the given ODE:

$$xy + y' + xy'' = \sum_n c_n(x-1)^n,$$

in which c_n is the sum of the terms in the rightmost column above, namely,

$$c_n = (n+2)(n+1)a_{n+2} + (n+1)^2a_{n+1} + a_n + a_{n-1}.$$

But to satisfy (†), we need $c_n = 0$ for each n, i.e.,

$$(n+2)(n+1)a_{n+2} + (n+1)^2a_{n+1} + a_n + a_{n-1} = 0.$$

For $n \geq 0$ this gives the recurrence relation

$$a_{n+2} = -\frac{(n+1)^2a_{n+1} + a_n + a_{n-1}}{(n+2)(n+1)}. \qquad (R)$$

Choose $a_0 = 1, a_1 = 0$ to get one solution using (R), recalling that $a_{-1} = 0$:

$$\begin{aligned}
n=0: &\qquad a_2 = -\frac{a_1 + a_0 + 0}{(2)(1)} = -\frac{1}{2},\\
n=1: &\qquad a_3 = -\frac{4a_2 + a_1 + a_0}{(3)(2)} = -\frac{-2+0+1}{(3)(2)} = \frac{1}{6},\\
n=2: &\qquad a_4 = -\frac{9a_3 + a_2 + a_1}{(4)(3)} = \cdots = -\frac{1}{12},\ \text{etc.}
\end{aligned}$$

This gives one solution,

$$y_1(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{12}(x-1)^5 + \cdots.$$

Choose $a_0 = 0, a_1 = 1$ to get a second solution using (R), again using $a_{-1} = 0$:

$$\begin{aligned}
n = 0: &\qquad a_2 = -\frac{a_1 + a_0 + 0}{(2)(1)} = -\frac{1}{2},\\
n = 1: &\qquad a_3 = -\frac{4a_2 + a_1 + a_0}{(3)(2)} = -\frac{-2+0+1}{(3)(2)} = \frac{1}{6},\\
n = 2: &\qquad a_4 = -\frac{9a_3 + a_2 + a_1}{(4)(3)} = \cdots = -\frac{1}{6}, \text{ etc.}
\end{aligned}$$

This gives the solution

$$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots.$$

Convergence Discussion (optional). Equation (†) is Bessel's equation of order 0, so its general solution has the form $y(x) = c_1 J_0(x) + c_2 Y_0(x)$. Both y_1 and y_2 found above must therefore be expressible in this form for suitable (different) choices of c_1, c_2. In fact,

$$\begin{aligned}
y_1(x) &= \left(\frac{Y_1(1)}{J_1(1)Y_0(1) - Y_1(1)J_0(1)}\right) J_0(x) + \left(\frac{J_1(1)}{J_1(1)Y_0(1) - Y_1(1)J_0(1)}\right) Y_0(x),\\
y_2(x) &= \left(\frac{Y_0(1)}{J_1(1)Y_0(1) - Y_1(1)J_0(1)}\right) J_0(x) + \left(\frac{J_0(1)}{J_1(1)Y_0(1) - Y_1(1)J_0(1)}\right) Y_0(x).
\end{aligned}$$

Both solutions contain a nonzero multiple of $Y_0(x)$, so both diverge as $x \to 0^+$, and this confirms that the radius of convergence for both series is exactly 1. ////

Alternative — Change of Variable. Defining a new variable $t = x - x_0$ transforms the statement "x is near x_0" into the statement "t is near 0". This substitution turns the general Taylor-style series $\sum_k a_k(x-x_0)^k$ into the Maclaurin-style $\sum_k a_k t^k$, with the same coefficients a_k. Let's illustrate these observations, with $x_0 = 1$, in the example of

$$xy'' + y' + xy = 0. \tag{†}$$

First, use the substitution $x = 1+t$ to replace all x-values, including the ones "inside" the functions y, y', y'':

$$(1+t)y''(1+t) + y'(1+t) + (1+t)y(1+t) = 0.$$

Then introduce a new unknown function w, by defining $w(t) = y(1+t)$. According to the Chain Rule, $w'(t) = y'(1+t)$ and $w''(t) = y''(1+t)$, so the ODE satisfied by w is

$$(1+t)w''(t) + w'(t) + (1+t)w(t) = 0. \tag{‡}$$

Now we can find a series solution of the form $w(t) = \sum a_k t^k$ for (‡). The big advantage here is that coefficient functions of $(1+t)$ attached to w and w'' are already expressed as power series in the variable t. The change of variables proposed here takes the place of the trick "write $x = 1 + (x - 1)$" used earlier.

Steps almost identical to the ones shown above lead to two independent power series solutions for (‡):

$$\begin{aligned} w_1(t) &= 1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^4 + \frac{1}{12}t^5 + \cdots, \\ w_2(t) &= t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{6}t^4 + \frac{3}{20}t^5 + \cdots. \end{aligned}$$

To recover solutions to the original problem, invert the substitution $x = 1 + t$ to get $t = x - 1$, and remember $y(x) = y(1 + t) = w(t) = w(x - 1)$. Thus, as before,

$$\begin{aligned} y_1(x) &= w_1(x-1) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{12}(x-1)^5 + \cdots, \\ y_2(x) &= w_2(x-1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots. \end{aligned}$$

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Initial Conditions and the Expansion Centre. Suppose the power series below has radius of convergence $\rho > 0$:

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots.$$

Then, as noted above,

$$y'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1} = 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \ldots.$$

Evaluation at the expansion centre $x = x_0$ is particularly easy, because each series has at most one nonzero term:

$$y(x_0) = a_0, \qquad y'(x_0) = a_1.$$

There are several ways to think about the implications of this fact.

1. For an initial-value problem like

$$(9 + x^2)y'' + 3xy' + 7y = 0, \qquad y(4) = 257,\ y'(4) = 316,$$

   it's wise to choose $x_0 = 4$ as the expansion centre. With this choice, the series form $y = \sum_{k=0}^{\infty} a_k (x - 4)^k$ fits the general discussion above, so the given initial conditions directly imply $a_0 = 257$ and $a_1 = 316$. The recurrence relation will produce definite numbers for $a_2, a_3, \ldots$, and the resulting series will be the unique solution to the stated initial-value problem.

By contrast, imagine trying a solution of the form $y = \sum_{k=0}^{\infty} \alpha_k x^k$ for the problem above. (Here the coefficients are named α_k because they are almost certain to be different from the a_k's for the series centred at 4.) There would be no obvious way to determine the constants α_0 and α_1. The naive approach might be to try to enforce the conditions

$$257 = \sum_{k=0}^{\infty} 4^k \alpha_k, \qquad 316 = \sum_{k=1}^{\infty} k 4^{k-1} \alpha_k.$$

On the surface, this looks difficult because it would require coming up with some reliable estimates of the series on the right in terms of α_0 and α_1. At a deeper level, note that the series centred at 0 will have radius of convergence 3, so plugging in $x = 4$ will not just be unpleasant ... it will be logically undefined!

2. Specifying values for $y(x_0)$ and $y'(x_0)$ is enough to select a unique solution for the differential equation in a certain open interval centred at x_0.

3. It's always possible to express every coefficient a_k for $k \geq 2$ as a linear combination of a_0 and a_1, and then by factoring to write the general series solution as

$$y = a_0 y_1(x) + a_1 y_2(x) \tag{$*$}$$

for suitable solution functions y_1, y_2. When the constants a_0, a_1 are allowed to take arbitrary real values, line $(*)$ provides the general solution of the given ODE.

Fine Points of Factorial Notation (Optional). Recall the convention that $0! = 1$. For descending products of odd numbers, some writers use the doubled exclamation point notation

$$1 \times 3 \times 5 \times \cdots \times (2n-1) = (2n-1)!!.$$

Descending powers of even numbers can be rewritten as follows:

$$2 \times 4 \times 6 \times 8 \times \cdots \times (2n) = 2^n \left(1 \times 2 \times 3 \times \cdots n\right) = (2^n) n!.$$

B. Singular Points

The standard-form equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

has a "singular point" at x_0 if one of p or q (or both) fails to be analytic at x_0. Careful study of precisely how analyticity fails lets us identify a sub-family of singular points where a systematic series-based approach is still useful. As we will soon see, we can overcome a singularity at x_0 if both bracketed functions below are analytic at x_0:

$$(x-x_0)^2 y'' + \left[(x-x_0)p(x)\right](x-x_0)y' + \left[(x-x_0)^2 q(x)\right] y = 0 \tag{1'}$$

This condition, "both bracketed functions are analytic at x_0", is the *definition* of *regular* for a singular point. Some singular points are regular, some are not, and it's important to be able to tell one kind from the other.

To understand (and remember) the defining test for regularity of a singular point x_0, think of ($1'$) as an *analytic perturbation of an Euler-type equation.* Obtain the *approximate Euler equation* by replacing the bracketed terms in ($1'$) with their limits at x_0:

$$\begin{aligned} &(x-x_0)^2y'' + B(x-x_0)y' + Cy = 0, \\ \text{where}\quad &B = \lim_{x\to x_0}\left[(x-x_0)p(x)\right], \quad C = \lim_{x\to x_0}\left[(x-x_0)^2q(x)\right]. \end{aligned} \tag{2}$$

We know how to solve (2): guess $y = (x-x_0)^s$ for some constant s. Then, since (1) is an analytic perturbation of (2), we can expect solutions for (1) to be analytic perturbations of solutions for (2), i.e., to have the form

$$y = (x-x_0)^s\left[1 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\right]. \tag{3}$$

Typically there are two values of s that work, and there are two ways to find them:

- □ analyze the pre-perturbed Euler equation (2), or
- □ tackle equation (1) directly, with a postulate of the form (3), and then insist on $a_0 = 1$ in the recurrence relation that comes out.

Notes. 1. The same constant term $a_0 = 1$ shows up in both linearly independent solutions. (This is quite different from the situation for "ordinary points".)

2. The radius of convergence for the series factor in (3) is at least the distance from x_0 to nearest singularity in $\mathbb{C}$ for the bracketed expressions $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$.

3. The series forms in (3) are subject to "exponent collisions" if the relevant s-values are identical, or separated by a positive integer. That requires separate study (later). For now, apply the methods here to the larger s-value to get one solution; it's only the search for the second solution where things get tricky. (This concern arises only when $s_1 - s_2$ is an integer; outside this case, both roots give solutions by the same method.)

Generic Example. Discuss $2xy'' + y' + xy = 0$.

Solution. Rewrite in standard form:

$$y'' + \frac{1}{2x}y' + \frac{1}{2}y = 0.$$

Note bad behaviour at 0 for coefficient of y': deduce that $x = 0$ is a singular point.

Multiply by x^2 and group factors carefully to express the same equation in perturbed-Euler format:

$$x^2y'' + \left[\frac{1}{2}\right]xy' + \left[\frac{x^2}{2}\right]y = 0.$$

If bracketed terms were constants, this would have Euler type. Inside the brackets are functions analytic at $x=0$, so this singular point is *regular.* Pre-perturbation Euler equation would be

$$x^2y'' + \tfrac{1}{2}xy' + 0y = 0,$$

with solutions of form $y = x^s$ when $0 = s(s-1) + \frac{1}{2}s = s(s-\frac{1}{2})$, i.e., $s = 0, \frac{1}{2}$. The quadratic equation $s^2 - \frac{1}{2}s = 0$ is called the *indicial equation*; the roots $s = 0, \frac{1}{2}$ are called the *exponents of singularity.* At this point the approximate Euler equation has given us all it can, namely, the values $s = 0$ and $s = \frac{1}{2}$. We return to the original equation.

Solutions will have form

$$y = x^s \sum_k a_k x^k = \sum_k a_k x^{k+s}, \qquad a_0 = 1 \quad (\text{and } a_k = 0 \text{ for } k < 0).$$

Note $y' = \sum_k (k+s)a_k x^{k+s-1}$, $y'' = \sum_k (k+s)(k+s-1)a_k x^{k+s-2}$. Inspired by the full original ODE, we deduce

$$\begin{aligned}
x^2y'' &= \sum_k (k+s)(k+s-1)a_k x^{k+s} = \sum_n (n+s)(n+s-1)a_n x^{n+s} \\
\tfrac{1}{2}xy' &= \sum_k \tfrac{1}{2}(k+s)a_k x^{k+s} = \sum_n \tfrac{1}{2}(n+s)a_n x^{n+s} \\
\frac{x^2}{2}y &= \sum_k \tfrac{1}{2}a_k x^{k+s+2} = \sum_n \tfrac{1}{2}a_{n-2}x^{n+s}
\end{aligned}$$

The given ODE requires

$$\begin{aligned}
0 &= x^2y'' + \left[\frac{1}{2}\right]xy' + \left[\frac{x^2}{2}\right]y \\
&= \sum_n \left[(n+s)(n+s-1)a_n + \tfrac{1}{2}(n+s)a_n + \tfrac{1}{2}a_{n-2}\right]x^{n+s} \\
&= x^s \sum_n \left[(n+s)(n+s-\tfrac{1}{2})a_n + \tfrac{1}{2}a_{n-2}\right]x^n.
\end{aligned}$$

By the identity theorem, this requires

$$(n+s)(n+s-\tfrac{1}{2})a_n = -\tfrac{1}{2}a_{n-2}, \qquad n \in \mathbb{Z}. \tag{$*$}$$

Notice that plugging in $n = 0$ and using $a_0 = 1$ (with $a_{-2} = 0$), we get the *same indicial equation as before,*

$$s(s-\tfrac{1}{2}) = 0, \qquad \text{i.e.,} \qquad s = 0,\ s = \tfrac{1}{2}.$$

(This provides a nice independent confirmation of the analysis above.) Then we use $(*)$ for all $n > 0$:

$$a_n = -\frac{1}{2(n+s)(n+s-\frac{1}{2})}a_{n-2}, \qquad n \geq 1.$$

Case $s = \frac{1}{2}$: $a_n = -\dfrac{1}{2(n+\frac{1}{2})n} a_{n-2} = -\dfrac{a_{n-2}}{n(2n+1)}$. Hence

$$\begin{aligned}
&a_1 = 0,\ a_3 = 0,\ a_5 = 0,\ \ldots, \\
&a_2 = -\frac{1}{2(5)} a_0 = -\frac{1}{2(5)}, \\
&a_4 = -\frac{1}{4(9)} a_2 = \frac{(-1)^2}{[4\times 2](9\times 5)}, \\
&a_6 = -\frac{1}{6(13)} a_4 = \frac{(-1)^3}{[6\times 4\times 2](13\times 9\times 5)}, \\
&\quad\vdots
\end{aligned}$$

This gives $y_1 = \sqrt{x}\left[1 - \dfrac{x^2}{10} + \dfrac{x^4}{360} \mp \cdots\right]$.

Case $s = 0$: $a_n = -\dfrac{1}{2n(n-\frac{1}{2})} a_{n-2} = -\dfrac{a_{n-2}}{n(2n-1)}$. Hence

$$\begin{aligned}
&a_1 = 0,\ a_3 = 0,\ a_5 = 0,\ \ldots; \\
&a_2 = -\frac{1}{2(3)} a_0 = -\frac{1}{2(3)}, \\
&a_4 = -\frac{1}{4(7)} a_2 = \frac{(-1)^2}{[4\times 2](7\times 3)} \\
&a_6 = -\frac{1}{6(11)} a_4 = \frac{(-1)^3}{[6\times 4\times 2](11\times 7\times 3)}, \\
&\quad\vdots
\end{aligned}$$

This gives $y_2 = 1 - \dfrac{x^2}{6} + \dfrac{x^4}{168} \mp \cdots$. ////

Challenge Problem—BDP 5.7 #14 in some previous edition. Find the form of two Frobenius-style solutions around $x = 1$:

$$(\ln x)y'' + \tfrac{1}{2}y' + y = 0.$$

Solution. Focus on $x > 1$, using the substitution $x = 1 + t$. The given ODE requires

$$\ln(1+t)y''(1+t) + \tfrac{1}{2}y'(1+t) + y(1+t) = 0.$$

Introduce the new function $u(t) = y(1+t)$, noting that $u'(t) = y'(1+t)$ and $u''(t) = y''(1+t)$, so

$$\ln(1+t)u''(t) + \tfrac{1}{2}u'(t) + u(t), \qquad t > 0. \tag{$*$}$$

We will solve this ODE for $u(t)$, then reverse the steps in the substitution:

$$y(x) = y(1+t) = u(t) = u(x-1).$$

Let's start.

Divide $(*)$ by $\ln(1+t)$ to see that $t=0$ is a singular point. Then multiply by t^2 and rearrange to get

$$t^2u''(t)+\left[\frac{t}{2\ln(1+t)}\right]tu'(t)+\left[\frac{t^2}{\ln(1+t)}\right]u(t)=0.$$

If bracketed terms were constant, we would have an Euler equation. At least those bracketed terms are analytic, since (whenever $|t|<1$)

$$\begin{aligned}
\ln(1+t)&=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}\pm\cdots,\\
\frac{t}{2\ln(1+t)}&=\frac{t}{2\left(t-t^2/2+t^3/3\mp\cdots\right)}=\frac{1}{2\left(1-t/2+t^2/3\mp\cdots\right)},\\
\frac{t^2}{\ln(1+t)}&=\frac{t^2}{\left(t-t^2/2+t^3/3\mp\cdots\right)}=\frac{t}{2\left(1-t/2+t^2/3\mp\cdots\right)}.
\end{aligned}$$

Taking limits as $t\to 0$ inside brackets only gives

$$t^2u''+\tfrac{1}{2}tu'+0u=0,\quad \text{so } u=t^s \text{ solves iff}\quad 0=s(s-1)+\tfrac{1}{2}s=s(s-\tfrac{1}{2}).$$

Thus we will have two solutions of basic Frobenius type,

$$\begin{aligned}
u_1&=\sqrt{t}\sum_{k=0}^{\infty}a_kt^k=\sum_{k=0}^{\infty}a_kt^{k+\frac{1}{2}},\qquad a_0=1,\\
u_2&=t^0\sum_{k=0}^{\infty}a_kt^k,\qquad a_0=1.
\end{aligned}$$

Calculations, heavy at times, give

$$\begin{aligned}
u_1(t)&=\sqrt{t}[1-\frac{3}{4}t+\frac{53}{480}t^2+\cdots],\quad y_1(x)=\sqrt{x-1}[1-\frac{3}{4}(x-1)+\frac{53}{480}(x-1)^2+\cdots].\\
u_2(t)&=\left[1-2t+\frac{2}{3}t^2+\cdots\right],\quad y_2(x)=\left[1-2(x-1)+\frac{2}{3}(x-1)^2+\cdots\right].
\end{aligned}$$

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Core Elements of These Problems. For a given ODE with a singular point x_0, the basic issues are

(i) Is the singular point x_0 regular?

(ii) If so, what are the exponents of singularity?

Formulating the approximate Euler equation that answers these two questions is an important skill that reveals lots of useful information before any terms of the series solution are calculated. Some exam and homework questions ask these two questions only. Strategic advice: Don't find the series unless you are sure it is needed!

Example. [Something from a 2013w2 midterm.]

C. Repeated Indicial Roots

Imagine that $x = 0$ is a regular singular point for the generic equation

$$y'' + p(x)y' + q(x)y = 0,$$

and that the indicial equation gives two identical real roots: $s_1 = s_2 = s$. Using this s in the standard approach above will produce a solution y_1 just as above. But we expect every second-order ODE to have two linearly independent solutions. What is the second one? Analogy with the pure Euler situation makes the form below plausible (see text for firmer reasons to use it):

$$y_2(x) = \ln(x)y_1(x) + x^s \sum_{k=1}^{\infty} b_k x^k = x^s \ln(x) \sum_{k=0}^{\infty} a_k x^k + x^s \sum_{k=1}^{\infty} b_k x^k.$$

The first term involves the series solution y_1 that we have presumably already found; the constants a_k are all known. The extra series is new, and we find the additional unknowns $b_1, b_2, \ldots$ by plugging it into the original equation and finding a recurrence relation between the b_k's. Note that the second series starts with $k = 1$, so the lowest power of x appearing there is actually x^{s+1}. If we were to rearrange the terms in both solutions in decreasing order of size near $x = 0$, we would have

$$\begin{aligned} y_1(x) &= x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots, \\ y_2(x) &= x^s \ln(x) + a_1 x^{s+1} \ln(x) + b_1 x^{s+1} + a_2 x^{s+2} \ln(x) + b_2 x^{s+2} + \ldots. \end{aligned}$$

In UBC Math 257/316, students may be expected to remember the form of $y_2(x)$, but actually calculating the constants b_k is considered slightly beyond the scope of the course.

Example [December 1990 Final Exam, Question 1]. Discuss

$$xy'' + (1 - x^2)y' - 2xy = 0.$$

Discussion. This ODE is singular at $x = 0$. Reorganize it as

$$x^2 y'' + [1 - x^2]xy' + [-2x^2]y = 0,$$

to see that the singular point at $x = 0$ is regular. The approximate Euler equation is

$$x^2 y'' + xy' = 0.$$

Guessing $y = x^s$ leads to the indicial equation $s(s - 1) + s = 0$, i.e., $s^2 = 0$. So let $s_1 = 0$, $s_2 = 0$. A first solution to the full original ODE will have the form

$$y = \sum_k a_k x^{k+s} \qquad [a_0 = 1]:$$

$$\begin{aligned}
0 &= -2x^2y + xy' - x^3y' + x^2y'' \\
&= \sum_k [-2]a_k x^{k+s+2} + \sum_k [k+s]a_k x^{k+s} + \sum_k [-(k+s)]a_k x^{k+s+2} \\
&\qquad\qquad\qquad + \sum_k (k+s)(k+s-1)a_k x^{k+s} \\
&= \sum_n [-2]a_{n-2} x^{n+s} + \sum_n [n+s]a_n x^{n+s} + \sum_n [-(n-2+s)]a_{n-2} x^{n+s} \\
&\qquad\qquad\qquad + \sum_n (n+s)(n+s-1)a_n x^{n+s} \\
&= \sum_n \left[-2a_{n-2} + (n+s)a_n - (n-2+s)a_{n-2} + (n+s)(n+s-1)a_n \right] x^{n+s} \\
&= \sum_n \left[(n+s)^2 a_n - (n+s)a_{n-2} \right] x^{n+s}.
\end{aligned}$$

Equate coefficients, cancel lightly, get

$$(n+s)a_n = a_{n-2}, \qquad \text{all } n \geq 0.$$

When $n = 0$, $a_0 = 1$ so this requires $s = a_{-2} = 0$, consistent with our earlier findings. (Good.) So use $s = 0$ and go forward:

$$a_n = \frac{1}{n} a_{n-2}, \qquad \text{all } n \geq 1.$$

This gives $a_1 = a_{-1} = 0$, hence $a_k = 0$ for all odd k. Meanwhile,

$$\begin{aligned}
a_2 &= \frac{1}{2} a_0 = \frac{1}{2} = \frac{1}{2^1 1!}, \\
a_4 &= \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} = \frac{1}{2^2 2!}, \\
a_6 &= \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} = \frac{1}{2^3 3!}, \ \cdots \\
a_{2k} &= \frac{1}{2^k k!}.
\end{aligned}$$

So one solution is

$$y_1(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = e^{x^2/2}.$$

For a second solution, since $s_2 = s_1 = 0$, discussion above suggests the form

$$\begin{aligned}
y_2(x) &= \ln(x) y_1(x) + x^s \left[b_1 x + b_2 x^2 + b_3 x^3 + \cdots \right] \\
&= \ln(x) \left[1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots \right] + b_1 x + b_2 x^2 + b_3 x^3 + \cdots.
\end{aligned}$$

Let's try for just b_1, b_2, by plugging into the original ODE: [nightmare of calculation] we get $b_1 = 0$, $b_2 = -1/4$. Maple gives the answer in this form:

$$y_2(x) = \ln(x)\left[1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + O\left(x^6\right)\right] + \left[-\frac{1}{4}x^2 - \frac{3}{32}x^4 + O\left(x^6\right)\right].$$

D. Indicial Roots Separated by a Positive Integer

Imagine that $x = 0$ is a regular singular point for the generic equation

$$y'' + p(x)y' + q(x)y = 0,$$

and that the roots of the indicial equation are s_1 and s_2, with $s_1 - s_2 = N$, a positive integer. The larger root gives a first solution in the usual way:

$$y_1(x) = x^{s_1} \sum_{k=0}^{\infty} a_k x^k, \qquad a_0 = 1 \quad (\text{and } a_k = 0 \text{ for } k < 0).$$

The form of a second solution for $x > 0$ is almost the same, except that an extra term appears:

$$y_2(x) = x^{s_2} \sum_{k=0}^{\infty} b_k x^k + \alpha \ln(x) y_1(x), \qquad b_0 = 1 \quad (\text{and } b_k = 0 \text{ for } k < 0).$$

Here α is a constant which, like the b_k's, must be found by postulating the form for y_2 shown here and plugging this whole expression back into the ODE to determine the constants.

In UBC Math 257/316, students may be expected to remember the form of $y_2(x)$, but actually calculating the constants α and b_k is considered beyond the scope of the course.

E. Bessel Equations (Optional)

Lots of researchers in past centuries got their names attached to differential equations. Equations useful in Physics, Chemistry, and Engineering are particularly popular. A classic example is Bessel's Equation of order n (for $n \geq 0$ any real number):

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

This has a regular singular point at $x = 0$, where the exponents of singularity come from the approximate Euler equation

$$x^2y'' + xy' - n^2y = 0.$$

Guessing $y = x^s$ here gives $s(s-1)+s-n^2 = 0$, i.e., $s^2 = n^2$, so $s = \pm n$. Since Bessel's Equation actually has important uses in mathematical applications, the study of

ODE's in which the indicial equation has roots separated by an integer has real-world significance.

In case $n = 0$, we have a repeated root at $s = 0$, which gives rise to one solution $y_1 = 1 + a_1 x + a_2 x^2 + \cdots$ analytic everywhere in $\mathbb{C}$, and a second solution whose general form is

$$y_2 = y_1(x)\ln(x) + b_1 x + b_2 x^2 + \cdots.$$

A suitable constant multiple of y_1 is used to define a bounded function known worldwide as J_0; some combination of y_1 and y_2 gives the other solution Y_0, for which there is a "logarithmic singularity" at $x = 0$.

When the constant n in Bessel's ODE is a positive integer, the indicial roots $s = \pm n$ are separated by an integer. The larger root gives a power series solution valid in the whole complex plane, of the form

$$J_n(x) = C_n x^n \left[1 + a_1 x + a_2 x^2 + \cdots\right].$$

The smaller root produces a solution $Y_n(x)$ that [mis]behaves like x^{-n} near the origin.

The functions J_n and Y_n are famous standard functions that scientists and engineers use with the same relaxed familiarity as sines, cosines, and exponentials. Even Microsoft Excel (with the "Analysis TookPak") can evaluate $J_n(x)$: the function of interest is `=BESSELJ(x,n)`. This makes Bessel's equation a legitimate target that could serve as a desirable simple outcome in some change-of-variables scheme.