Separation of Variables

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A. Three Famous PDE's

1. Wave Equation. Displacement u depends on position and time: $u = u(x,t)$. Concavity drives acceleration:

$$u_{tt} = c^2 u_{xx}.$$

2. Heat Equation. Temperature u depends on position and time: $u = u(x,t)$. Concavity drives flow rate:

$$u_t = \alpha^2 u_{xx}.$$

3. Laplace's Equation. Potential u depends on position in $2D$: $u = u(x,y)$. Concavity averages to 0:

$$u_{xx} + u_{yy} = 0.$$

All three equations share two important properties:

- **Homogeneity:** If u satisfies the equation, then ku does too, for all real k.
- **Linearity:** Superposition works. That is, if $u^{(1)}, \ldots, u^{(N)}$ are functions that satisfy the equation and constants $c_1, \ldots, c_N$ are given, then the function

  $$u \overset{\text{def}}{=} c_1 u^{(1)} + c_2 u^{(2)} + \cdots + c_N u^{(N)}$$

  satisfies the equation too.

B. Separation of Variables

For homogeneous linear PDE's, the following two-stage approach often works:

1. Identify simple product-form solutions.
2. Use linear superposition to combine these as needed.

We investigate idea 1 further in this section; idea 2 will help us later.

Separation in the PDE. Consider case $\alpha = 1$ of the heat equation:

$$u_t = u_{xx}. \tag{1}$$

Seek nonzero solutions of form "mode shape" × "amplitude factor", $u(x,t) = X(x)T(t)$. Which function pairs $X(x)$ and $T(t)$ will work?

$$u_t = u_{xx} \iff X(x)T'(t) = X''(x)T(t) \iff \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \qquad \text{all } t, \text{ all } x.$$

The function on the left in this identity depends only on the single variable t. However, changing the value of t makes no difference at all on the right side. The only functions of t that have no actual t-dependence are the constant functions. So the function of t on the left side must be some constant. Let's give this so-called "separation constant" a name. A typical choice is $-\lambda$. The minus sign will be convenient later, but it does not hide any kind of assumption about the sign of the separation constant. The expression $-\lambda$ could be negative, zero, or positive, depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. Anything can happen. Symbolically, our identity becomes

$$\frac{T'(t)}{T(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

This gives two ODE's, linked by their shared constant λ:

(i) $T'(t) = -\lambda T(t)$, all t,

(ii) $X''(x) + \lambda X(x) = 0$, all x.

Equation (i) holds if and only if $T(t) = Ce^{-\lambda t}$ for some constant C; in (ii), the form of the general solution is determined by the sign of λ. Three possibilities arise.

Case $\lambda < 0$. When $\lambda < 0$, we have $-\lambda > 0$, so $\sqrt{-\lambda}$ is a well-defined positive number. The general solution for (ii) is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}, \qquad A, B \in \mathbb{R}.$$

For any choice of $\lambda < 0$, this produces many solutions for (1), namely

$$u(x,t) = Ce^{-\lambda t}\left(Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}\right), \qquad A, B, C \in \mathbb{R}.$$

No generality is lost if we choose $C = 1$ and allow arbitrary choices for A and B.

Case $\lambda = 0$. Here the general solution for (ii) is $X(x) = Ax + B$, so another solution for (1) is

$$u(x,t) = Ce^{0t}(Ax + B) = C(Ax + B), \qquad A, B, C \in \mathbb{R}.$$

Again, no generality is lost if we choose $C = 1$ and allow arbitrary $A, B \in \mathbb{R}$.

Case $\lambda > 0$. When $\lambda > 0$, $\sqrt{\lambda}$ is a well-defined positive number. The general solution for (ii) is $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, leading to many more solutions for (1):

$$u(x,t) = Ce^{-\lambda t}\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right), \qquad A, B, C \in \mathbb{R}.$$

Again, choosing $C = 1$ does not limit the variety of choices listed here.

Examples. By choosing different values for λ, A, and B in the cases considered above, we can construct various solutions for the heat equation (1). Here are four possibilities; there are infinitely many others:

- $u^{(1)}(x,t) = 1 + x$

  (from $\lambda = 0$) – a straight line of slope 1 persists for all time.

- $u^{(2)}(x,t) = e^t\left(e^x + e^{-x}\right)$

(from $\lambda = -1$) – hyperbolic cosine profile in x grows exponentially as time advances.

- $u^{(3)}(x,t) = e^{-\pi^2 t}\sin(\pi x)$

  (from $\lambda = \pi^2$) – a sinusoidal profile in x decays exponentially as time advances.

- $u(x,t) = 1 + x - 12e^t\left(e^x + e^{-x}\right) + 2e^{-\pi^2 t}\sin(\pi x)$

  ($u = u^{(1)} - 12u^{(2)} + 2u^{(3)}$) – any linear combination of solutions is another solution, because the PDE is linear and homogeneous. Although we can describe the simple behaviour of each term, the overall function u is not easy to summarize. Note also that u is **not** of the simple separated form $X(x)T(t)$, even though it comes from a sum of elements with that special structure.

Fundamental Solution (opt). For practice, please check by differentiation that the function below also satisfies (1):

$$u(x,t) = \frac{1}{\sqrt{t}}e^{-x^2/4t}.$$

(You should find that the product rule gives

$$u_t = \left(\frac{-1}{2t^{3/2}} + \frac{x^2}{4t^{5/2}}\right)e^{-x^2/4t} = \frac{1}{4t^{5/2}}\left(x^2 - 2t\right)e^{-x^2/4t},$$

and then confirm that this coincides with u_{xx}.)

No General Solution. The heat equation has a dazzling variety of solutions in separated form (any real number λ produces infinitely many, because you can still adjust A and B any way you like), plus an interesting exponential solution (not of separated form), and all the possible superposition combinations built from these. There is no simple way to catalogue the collection of all solutions. In the world of partial differential equations, there is no meaningful concept of a "general solution" like we have for linear ODE's.

Homogeneous Boundary Conditions. The physical problems that lead to one of the three famous PDE's listed above often include information about the function u at the boundary of the spatial region of interest. Suppose this region is simply a real interval, $a \le x \le b$. Typically there are constants c_0, c_1 (not both zero) and d_0, d_1 (not both zero) for which the desired solution u satisfies

$$\begin{aligned} c_0u(a,t) + c_1u_x(a,t) &= 0, \qquad t > 0,\\ d_0u(b,t) + d_1u_x(b,t) &= 0, \qquad t > 0. \end{aligned} \tag{BC}$$

These requirements have the properties of linearity and homogeneity mentioned for the three famous PDE's in Section A. If we include them in the specifications of the product-form solutions $u(x,t) = X(x)T(t)$ considered in Section B, we must have

$$\begin{aligned} c_0X(a)T(t) + c_1X'(a)T(t) &= 0, \qquad t > 0,\\ d_0X(b)T(t) + d_1X'(b)T(t) &= 0, \qquad t > 0. \end{aligned}$$

To satisfy these conditions with a function $u(x,t) = X(x)T(t)$ that is not merely the constant 0, we must insist that neither factor ($X(x)$ or $T(t)$) is the constant 0. So the identity above, valid for all $t > 0$, reduces to a pair of boundary conditions on the factor $X(x)$:

$$c_0X(a) + c_1X'(a) = 0, \qquad d_0X(b) + d_1X'(b) = 0.$$

When combined with the ODE for X described in Section B above, these boundary conditions complete the specification of an eigenvalue problem. For later reference, let's write it out:

$$\begin{array}{lll} \text{(ODE)} & X''(x) + \lambda X(x) = 0, & a < x < b, \\ \text{BC(a)} & c_0X(a) + c_1X'(a) = 0, & \\ \text{BC(b)} & d_0X(b) + d_1X'(b) = 0. & \end{array} \qquad \text{(EVP)}$$

We have some experience with problems like this. When the interval $[a,b]$ happens to be $[0,\ell]$ for some $\ell > 0$, and when $(c_0,c_1) = (1,0)$ and $(d_0,d_1) = (1,0)$, we recognize the eigenvalue problem associated with the Fourier Sine Series. Different choices for (c_0,c_1) and (d_0,d_1) will produce other familiar problems from the Big Four.

Here's the point: in any particular situation, boundary information will show that many of the separated-form solutions $u(x,t) = X(x)T(t)$ for the PDE alone are actually irrelevant. Ignoring these lets us focus on the smaller collection these solutions where X is an eigenfunction in a problem like (EVP). Note, however, that different boundary conditions in the original problem will produce different boundary conditions in (EVP), so there is still plenty of room for variety in the solutions we must be prepared to work with.

Modes and Music. It's remarkable that the same eigenvalue problem (EVP) is useful in each of the famous PDE's in Section A. Here are the details:

1. A typical boundary-value problem for **the wave equation** involves

$$\begin{array}{llll} \text{(PDE)} & u_{tt} = c^2u_{xx}, & 0 < x < \ell, & t > 0, \\ \text{BC(0)} & c_0u(0,t) + c_1u_x(0,t) = 0, & & t > 0, \\ \text{BC}(\ell) & d_0u(\ell,t) + d_1u_x(\ell,t) = 0, & & t > 0. \end{array}$$

   In any nontrivial solution of separated form $u(x,t) = X(x)T(t)$, the factor $X(x)$ must be an eigenfunction for (EVP), and the corresponding eigenvalue λ links the solution $X(x)$ to the equation

$$T''(t) + \lambda c^2T(t) = 0.$$

   When $\lambda = \omega^2 > 0$, the eigenfunction $X(x)$ describes the shape of a physical response, like the displacement of a guitar string — what you *see*, while the time-dependent factor $T(t) = R\cos(\omega ct - \phi)$ describes a sinusoidal oscillation with angular frequence ωc — what you *hear*. Physical modes with larger values of ω correspond to oscillatory motions with correspondingly higher audible pitches ωc. A sample solution arising when (EVP) has FSS type could be

$$u(x,t) = \cos\Big(\frac{n\pi}{\ell} ct\Big) \sin\Big(\frac{n\pi x}{\ell}\Big) .$$

2. A typical boundary-value problem for **the heat equation** involves

$$\begin{array}{lll} \text{(PDE)} & u_t = \alpha^2 u_{xx}, & 0 < x < \ell, \quad t > 0, \\ \text{BC}(0) & c_0 u(0,t) + c_1 u_x(0,t) = 0, & t > 0, \\ \text{BC}(\ell) & d_0 u(\ell,t) + d_1 u_x(\ell,t) = 0, & t > 0. \end{array}$$

In any nontrivial solution of separated form $u(x,t) = X(x)T(t)$, the factor $X(x)$ must be an eigenfunction for (EVP), and the corresponding eigenvalue λ links the solution $X(x)$ to the equation

$$T'(t) + \lambda\alpha^2 T(t) = 0.$$

When $\lambda = \omega^2 > 0$, the eigenfunction $X(x)$ describes the shape of a physical response, like the temperature values along a pipe, while the time-dependent factor $T(t) = Ke^{-\alpha^2\omega^2 t}$ describes exponential decay. Physical modes with larger values of ω have decay-factors in which the rate constant $\alpha^2\omega^2$ is correspondingly larger. A sample solution arising when (EVP) has FSS type could be

$$u(x,t) = e^{-n^2\pi^2\alpha^2 t/\ell^2} \sin\Big(\frac{n\pi x}{\ell}\Big) .$$

3. A typical boundary-value problem for **Laplace's equation** involves

$$\begin{array}{lll} \text{(PDE)} & 0 = u_{xx} + u_{yy}, & 0 < x < \ell, \quad 0 < y < b, \\ \text{BC}(0) & c_0 u(0,y) + c_1 u_x(0,y) = 0, & 0 < y < b, \\ \text{BC}(\ell) & d_0 u(\ell,t) + d_1 u_x(\ell,t) = 0, & 0 < y < b. \end{array}$$

In any nontrivial solution of separated form $u(x,y) = X(x)Y(y)$, the factor $X(x)$ must be an eigenfunction for (EVP), and the corresponding eigenvalue λ links the solution $X(x)$ to the equation

$$Y''(y) - \lambda Y(y) = 0.$$

When $\lambda = \omega^2 > 0$, the eigenfunction $X(x)$ describes a sinusoidal dependence on the x-coordinate, while the perpendicular factor $Y(t) = Ae^{\omega y} + Be^{-\omega y}$ describes exponential dependence on y. In every such solution, the coefficients of y in the exponential factors must be proportional to the angular frequency ω in the sinusoidal direction. A sample solution arising when (EVP) has FSS type could be

$$u(x,y) = \Big(e^{n\pi y/\ell} - e^{-n\pi y/\ell}\Big) \sin\Big(\frac{n\pi x}{\ell}\Big) .$$

Notice how each representative eigenfunction from the FSS family attracts a different style of multiplicative factor in each of the classic problem types above. In the wave equation, FSS modes tend to oscillate; in the heat equation, they decay; and in the Laplace equation they are complemented by exponentials in the independent variable.

C. Generalized Superposition and Eigenfunction Series Solutions

We are already experienced with solving classic boundary-value problems by developing suitable **eigenfunction series with time-varying coefficients.** Separation of variables offers another explanation for this approach. Each term in the series we ultimately write down has the simple separated form discussed in earlier sections. The series is just an infinite superposition of these basic elements, with superposition coefficients chosen to respect given initial information.

Let's illustrate all this with three solved examples. The first one has commentary comparing the separation of variables approach to the 6-step recipe endorsed by your instructor.

A Heat Problem–FCS style.

(PDE) $u_t = \alpha^2 u_{xx}$, $0 < x < \pi$, $t > 0$,

(BC) $u_x(0,t) = 0,\ u_x(\pi,t) = 0$, $t > 0$,

(IC) $u(x,0) = f(x)$, $0 < x < \pi$.

Here $f(x) = \begin{cases} x, & \text{if } 0 < x \le \pi/2, \\ \pi - x, & \text{if } \pi/2 < x < \pi, \end{cases}$ as on the `triwav` handout. (That handout is on the course web page, coded RR and dated 11 Jun 2014.)

Eigen-analysis: Plugging $u(x,t) = X(x)T(t)$ into (PDE)–(BC) produces the following eigenvalue problem for the factor $X(x)$.

$$X''(x) + \lambda X(x) = 0,\ 0 < x < \pi; \qquad X'(0) = 0 = X'(\pi).$$

This one is familiar. We know that the eigenfunctions are associated with the Fourier Cosine Series. In detail, they are

$$X_n(x) = \cos(nx), \qquad n = 0, 1, 2, \ldots.$$

Separation of variables links each of these functions with an eigenvalue λ_n and a corresponding time-varying factor $T_n(t)$. The safest thing to do about λ_n and T_n at this stage is to ignore them, because they will emerge organically as we proceed.

Postulate: Assume that for some functions $T_n(t)$, the solution has the form

$$u(x,t) = \tfrac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(nx). \qquad (*)$$

(Indeed these $T_n(t)$ functions will be the same ones that we could have found using separation of variables, but at this stage we don't need to know that.)

Initialize: From (IC),

$$f(x) = u(x,0) = \tfrac{1}{2}T_0(0) + \sum_{n=1}^{\infty} T_n(0)\cos(nx), \qquad 0 < x < \pi.$$

That is, the numbers $T_n(0)$ are the FCS coefficients for the given function $f(x)$. From the handouts,

$$T_n(0) = \frac{2}{\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right],\ n \ge 1; \qquad T_0(0) = \frac{\pi}{2}. \qquad (**)$$

Propagate: The (PDE) reveals the t-dependence of the unknown functions $T_n(t)$.

$$0 = u_t - \alpha^2 u_{xx} = \tfrac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} \left[T_n'(t)\cos(nx) - \alpha^2 T_n(t)\left(-n^2\cos(nx)\right)\right]$$

$$= \tfrac{1}{2}\left[T_0'(t)\right] + \sum_{n=1}^{\infty}\left[T_n'(t) + n^2\alpha^2 T_n(t)\right]\cos(nx).$$

At each instant t, the RHS is a FCS. It gives an expansion for the **constant function 0**, whose FCS coefficients are all 0. Hence each expression in brackets must vanish:

$$T_n'(t) + n^2\alpha^2 T_n(t) = 0, \quad t > 0, \qquad n = 0, 1, 2, \ldots.$$

(Here, at last, is exactly the same ODE for function $T_n(t)$ that one would get from separation of variables, with the benefit of coefficients that automatically confirm the correct sign and magnitude for the corresponding eigenvalue λ_n. Confronting this ODE now instead of at the beginning makes it perfectly clear where the solution is supposed to fit into the the grand scheme of the original problem.) Solve this ODE to get $T_n(t) = A_n e^{-n^2 t}$ for some constant A_n. (The case $n = 0$ always needs a little special attention in FCS problems, but here the general expression $T_0(t) = A_0$ correctly captures the constant solution we would expect for the ODE $T_0'(t) = 0$.) Then $A_n = T_n(0)$, which we know from $(**)$ above, gives

$$T_n(t) = T_n(0)e^{-n^2\alpha^2 t}.$$

Report: We conclude that

$$\boxed{u(x,t) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]e^{-n^2\alpha^2 t}\cos(nx).}$$

The qualitative features here match the intuition suggested earlier. That is, each term in the series involves some constant multiplying a sinusoidal mode in space with an amplitude factor that decays exponentially with time, and the higher spatial modes have the faster decay rates.

A Wave Problem, FSS-style.

(PDE)	$u_{tt} = c^2 u_{xx}$,	$0 < x < \pi$,	$t > 0$,
(BC)	$u(0,t) = 0,\ u(\pi,t) = 0$,		$t > 0$,
(IC)	$u(x,0) = f(x)$,	$0 < x < \pi$,	
	$u_t(x,0) = g(x)$,	$0 < x < \pi$.	

Eigen-analysis: Plugging $u(x,t) = X(x)T(t)$ into (PDE)–(BC) leads to the following eigenvalue problem for the factor $X(x)$.

$$X''(x) + \lambda X(x) = 0,\ 0 < x < \pi; \qquad X(0) = 0 = X(\pi).$$

This is one we recognize: the full list of all eigenfunctions is precisly the basis for the Fourier Sine Series, namely,

$$X_n(x) = \sin(nx), \qquad n = 1, 2, \ldots.$$

At this point, separation of variables has done all it can for us. Ignore the eigenvalues and the ODE for the factor $T(t)$: all these will come up again automatically in later steps.

Postulate:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx). \tag{†}$$

Initialize: By (IC),

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin(nx) \implies T_n(0) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx,$$
$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) \sin(nx) \implies T_n'(0) = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)\, dx.$$

Propagate: Using (†) in (PDE) gives

$$0 = u_{tt} - c^2 u_{xx} = \sum_{n=1}^{\infty} \Big[T_n''(t) \sin(nx) - c^2 T_n(t) \left(-n^2 \sin nx \right) \Big]$$
$$= \sum_{n=1}^{\infty} \left[T_n''(t) + n^2 c^2 T_n(t) \right] \sin nx.$$

At each instant t, this is an equation about Fourier Sine Series with respect to x: the function on the left side is expressed by the right. But the function on the left is just the constant 0, so all the Fourier Sine coefficients must be 0 too: hence

$$T_n''(t) + n^2 c^2 T_n(t) = 0,$$

The general solution of this ODE is $T_n(t) = A_n \cos(nct) + B_n \sin(nct)$ (exactly as before), and the initial values found above determine the constants via

$$A_n = T_n(0) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx,$$
$$ncB_n = T_n'(0) = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)\, dx.$$

Report: When $g(x) = 0$ and $f(x) = \begin{cases} x, & \text{if } 0 < x \le \pi/2, \\ \pi - x, & \text{if } \pi/2 < x < \pi, \end{cases}$

$$\boxed{u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nct) \sin(nx).}$$

A Laplace Problem–HPSS style.

(PDE)	$u_{xx} + u_{yy} = 0,$	$0 < x < \pi,$	$0 < y < 2\pi,$
(BC_1)	$u(0,y) = 0,\ u_x(\pi,y) = 0,$		$0 < y < 2\pi,$
(BC_2)	$u(x,0) = f(x),\ u(x,2\pi) = g(x)$	$0 < x < \pi.$	

Here functions f and g are part of the problem statement.

Eigen-analysis: Separation of variables $u(x,y) = X(x)Y(y)$ in (PDE)–(BC_1) to a HPSS eigenproblem for $X(x)$ on $0 < x < \pi$. Discard the eigenvalues and the eigenfunctions

$$X_n(x) = \sin(\omega_n x)\,, \quad \omega_n \stackrel{\text{def}}{=} \frac{2n-1}{2}, \qquad n = 1, 2, \ldots.$$

Postulate $$u(x,t) = \sum_{n=1}^{\infty} Y_n(y) \sin(\omega_n x). \tag{$**$}$$

Initialize: by (BC), $\quad f(x) = u(x,0) = \sum_{n=1}^{\infty} Y_n(0) \sin(\omega_n x),\ 0 < x < \pi.$

Hence the constants $Y_n(0)$ must be the HPSS coefficients for function f:

$$Y_n(0) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(\omega_n x)\, dx. \tag{1}$$

Likewise, $\quad g(x) = u(x,2\pi) = \sum_{n=1}^{\infty} Y_n(2\pi) \sin(\omega_n x),\ 0 < x < \pi,$

so the numbers $Y_n(2\pi)$ must be the HPSS coefficients for g:

$$Y_n(2\pi) = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(\omega_n x)\, dx. \tag{2}$$

Lines (1)–(2) provide two algebraic equations for each coefficient function Y_n.

Propagate: To satisfy (PDE), the postulated series above must obey

$$\begin{aligned} 0 = u_{xx} + u_{yy} &= \sum_{n=1}^{\infty} \left[Y_n(y) \left(-\omega_n^2 \sin(nx) \right) + Y_n''(y) \sin(nx) \right] \\ &= \sum_{n=1}^{\infty} \left[Y_n''(t) - \omega_n^2 Y_n(y) \right] \sin(\omega_n x). \end{aligned}$$

For each fixed y, the RHS is an HPSS identity involving functions of x. It gives an expansion for the *constant function* 0, whose HPSS coefficients are all 0. Hence we must have

$$Y_n''(y) - \omega_n^2 Y_n(y) = 0, \qquad 0 < y < 2\pi.$$

Solve this ODE problem to get $Y_n(y) = A_n e^{\omega_n y} + B_n e^{-\omega_n y}$. Use the linear system

$$\begin{aligned} A_n + B_n &= Y_n(0) = \langle \text{a known integral} \rangle \\ A_n e^{(2n-1)\pi} + B_n e^{-(2n-1)\pi} &= Y_n(2\pi) = \langle \text{another known integral} \rangle \end{aligned}$$

to solve for A_n and B_n.

Report: The answer will then be

$$\boxed{u(x,y)=\sum_{n=1}^{\infty}\left[A_n e^{\left(\frac{2n-1}{2}\right)y}+B_n e^{-\left(\frac{2n-1}{2}\right)y}\right]\sin\left(\frac{2n-1}{2}x\right).}$$

When the top and bottom boundary functions are given by

$$f(x)=\begin{cases} x, & \text{if } 0<x\le \pi/2,\\ \pi-x, & \text{if } \pi/2<x<\pi, \end{cases} \qquad g(x)=0,$$

as on the `triwav` handout, the "known integrals" above come to $Y_n(0)=p_n$ and $Y_n(2\pi)=0$. Thus $B_n=-e^{2(2n-1)\pi}A_n$, giving

$$\begin{aligned} A_n &= \frac{p_n}{1-e^{2\pi(2n-1)}}=\frac{e^{-2(2n-1)\pi}p_n}{e^{-2(2n-1)\pi}-1},\\ B_n &= \frac{p_n}{1-e^{-2(2n-1)\pi}}. \end{aligned}$$

Back in the boxed answer above, we get

$$u(x,y)=\sum_{n=1}^{\infty}\frac{p_n}{1-e^{-2(2n-1)\pi}}\left[e^{-\left(\frac{2n-1}{2}\right)y}-e^{\left(\frac{2n-1}{2}\right)(y-4\pi)}\right]\sin\left(\frac{2n-1}{2}x\right).$$