Solving Linear Boundary-Value Problems

UBC M316 Lecture Notes by Philip D. Loewen

A linear boundary value problem involves a partial differential equation (PDE), some boundary conditions (BC), and some initial conditions (IC). In many cases, following the recipe below will produce a solution. (Complications surrounding one or more steps in the recipe are the main source of variety and interest in this course!) To be definite, assume the unknown function in the problem is called u.

1. Homogenize. All basic methods will fail unless the problem has a certain minimum number of homogeneous boundary conditions (i.e., conditions satisfied by the function $u = 0$). For wave and heat problems in one space dimension, you need a homogeneous boundary condition at each end of the physical medium; for potential problems in two space dimensions, you need a pair of homogeneous boundary conditions on opposite faces of the domain.

To enforce homogeneity, split the unknown u into a sum: $u = v + w$, where v is a function you will choose in this step, and w becomes the new unknown function you will find later. In choosing v, the absolute requirement is

(i) v must satisfy the given (nonhomogeneous) boundary conditions.

Since there are infinitely many functions v satisfying the BC's, an intelligent choice is required. The guiding principle is

(ii) Substituting $v + w$ for u in the original PDE should produce a manageable PDE for the new unknown w.

In the best cases, one can choose v as a solution to the original PDE: then the resulting PDE for w will be a homogeneous version of the original PDE for u. NOTE: When such a choice is possible, it even simplifies problems that start out with homogeneous BC's.

At the end of this step, you should know the function v, and the reduced problem you must solve for w. The w-problem will have the same type as the original problem, except that it involves homogeneous boundary conditions and a modified (ideally, homogeneous) partial differential equation.

$\mapsto$ If the given problem has homogeneous BC's and a homogeneous PDE to start with, the choice $v = 0$ does everything required above. The remaining steps in the recipe work perfectly, with $u = w$. (Straightforward separation of variables works too, but is not really much faster.)

2. Identify Suitable Eigenfunctions. The new unknown function w satisfies a certain PDE with homogeneous BC's. Take the homogeneous version of the PDE and separate variables to find an eigenvalue problem of the usual sort. Work out the eigenfunctions as explicitly as possible. If integral formulas for generalized Fourier series involving these eigenfunctions are not already known, work them out now. (Here the orthogonality relation is crucial.)

From now on, suppose the unknown is $u = u(x, t)$ and the eigenvalue problem is in the x-variable, with resulting eigenfunctions $X_n(x)$. These variable names could be different in your particular problem, but all the ideas line up exactly.

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3. Postulate. Write down a proposed solution of the form $w(x, t) = \sum T_n(t)X_n(x)$, using the eigenfunctions $X_n(x)$ from Step 2. (This sum must include *all* eigenfunctions X_n.) This reduces the job of finding $w(x, t)$ to the task of identifying the coefficient functions $T_n(t)$.

4. Initialize. The boundary-value problem for w involves two boundary conditions that were used to define the eigenfunctions X_n. These two will be satisfied by the series postulated above no matter what functions T_n you choose, so focus on the initial conditions or unused boundary conditions that have not yet been used. Use them to get information on the values of the coefficient functions T_n at certain points.

- For heat problems, the unused information is an initial condition specifying $w(x, 0)$. This function must equal $\sum T_n(0)X_n(x)$, so the initial values $T_n(0)$ must be the coefficients in a generalized Fourier series expansion for the known function $w(x, 0)$. This observation allows you to find $T_n(0)$.
- For wave problems, the unused information is a pair of initial conditions specifying both $w(x, 0)$ and $w_t(x, 0)$. As above, comparing these known functions with their representations as generalized Fourier series allows you to find both $T_n(0)$ and $T_n'(0)$ from the standard integral formulas.
- For potential problems (where the letters "T" and "t" would look better as "Y" and "y"), the unused information is a pair of boundary conditions. As above, substituting the assumed series into these gives information about some combination of $T_n(0)$ and $T_n'(0)$ from one boundary condition, and some combination of $T_n(\ell)$ and $T_n'(\ell)$ from the other.

5. Propagate. Plug the postulated series expansion into the PDE for w. For each fixed t, think of the resulting equation as an identity between generalized Fourier series, where the basis functions are $X_n(x)$. This time the standard coefficient formulas lead to a linear differential equation for each unknown function $T_n(t)$—a first-order equation in the case of a heat problem, and a second-order equation in the wave and potential cases. Solve each of these equations, using the results of Step 4 to determine the constants of integration you get.

6. Reconcile and Report. After Step 5, you should have detailed knowledge of each coefficient function $T_n(t)$. Put these into the form postulated in Step 3 to produce a complete solution for $w(x, t) = \sum T_n(t)X_n(x)$. Then, review Step 1. If the function u requested in the original problem is not exactly w, because $v \not\equiv 0$, be sure to report the answer $u = v + w$ as promised earlier.