I. Linear Ordinary Differential Equations

UBC M316 Lecture Notes by Philip D. Loewen

A. Homogeneous Linear ODE's Constant Coefficients

First-Order ODE's. Find the unknown function y, given that

$$by'(x) + cy(x) = 0, \qquad \forall x \in \mathbb{R}. \tag{$*$}$$

[Here b, c are real constants, with $b \neq 0$.] The obvious solution $y = 0$ is called the **trivial solution.** We seek nontrivial solutions.

Second-Order ODE's. Given constants $a, b, c \in \mathbb{R}$ [$a \neq 0$], consider this linear equation:

$$ay''(x) + by'(x) + cy(x) = 0, \qquad \forall x \in \mathbb{R}. \tag{$**$}$$

The obvious solution $y = 0$ is called the **trivial solution.** We seek nontrivial solutions, paying particular attention to their zeros.

Standard "Guess". $y = e^{sx}$, $s \in \mathbb{C}$ constant. This function solves ($**$) iff

$$0 = (as^2 + bs + c)e^{sx}.$$

Characteristic Equation. $\qquad as^2 + bs + c = 0.$

Characteristic roots. $\qquad s = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}.$

Case 1: Distinct Real Roots [$b^2 - 4ac > 0$]**.** Call these s_1 and s_2, with $s_1 < s_2$. Every nontrivial solution has the form

$$y = k_1 e^{s_1 x} + k_2 e^{s_2 x} = e^{s_1 x}\left[k_1 + k_2 e^{(s_2 - s_1)x}\right], \qquad k_1, k_2 \in \mathbb{R},$$

with vector $(k_1, k_2) \neq (0, 0)$. Nontrivial solutions y have at most one zero, since bracketed term is monotonic.

Case 2: Distinct Complex Roots [$b^2 - 4ac < 0$]**.** Call these $s = \sigma \pm i\omega$, with $\omega > 0$. Every nontrivial solution has form below, with vector $(k_1, k_2) \neq (0, 0)$:

$$y = e^{\sigma x}\left[k_1 \cos \omega x + k_2 \sin \omega x\right] = Ae^{\sigma x}\cos(\omega x - \phi),$$

$$\text{where} \quad A = \sqrt{k_1^2 + k_2^2}, \quad (\cos \phi, \sin \phi) = (k_1/A, k_2/A).$$

Nontrivial solutions y are periodic with period $2\pi/\omega$: each one has infinitely many distinct zeros.

Case 3: Repeated Real Root [$b^2 - 4ac = 0$]**.** Call it s. Every nontrivial solution has form

$$y = k_1 e^{sx} + k_2 x e^{sx} = e^{sx}\left[k_1 + k_2 x\right], \qquad k_1, k_2 \in \mathbb{R},$$

with vector $(k_1, k_2) \neq (0, 0)$. Nontrivial solutions y have at most one zero, since bracketed term is monotonic.

Higher-Order ODE's. So far we have discussed cases $n = 1$ and $n = 2$ of the general setup

$$c_0 y + c_1 y' + c_2 y'' + \cdots + c_n y^{(n)} = 0, \qquad x \in \mathbb{R}, \tag{$\dagger$}$$

where $c_0, \ldots, c_n$ are real constants with $c_n \neq 0$. Again we "guess" $y = e^{st}$, plug in, and find a solution if and only if the constant $s \in C$ obeys

$$c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n = 0.$$

Solve this for s, expecting n roots (possibly complex, possibly repeated) and n linearly independent solutions. In the simplest case where there are n distinct real roots $s_1, \ldots, s_n$, the general solution is

$$y = k_1 e^{s_1 x} + k_2 e^{s_2 x} + \cdots + k_n e^{s_n x}, \qquad k_1, k_2, \ldots, k_n \in \mathbb{R}.$$

If $s, \overline{s}$ are a pair of complex-conjugate roots, they are responsible for two solutions of the form exponential$\times$sinusoid, just as in the second-order case above. If s is a repeated root of multiplicity p, it is responsible for producing p independent contributions to the general solution: these turn out to be $e^{sx}, xe^{sx}, x^2 e^{sx}, \ldots, x^{p-1} e^{sx}$.

Systems of First-Order Equations. For a system of n first-order ODE's in n unknown functions $y_1, \ldots, y_n$, like

$$\begin{aligned}
y_1'(t) &= a_{11} y_1 + a_{12} y_2 + a_{13} y_3 + \cdots + a_{1n} y_n \\
y_2'(t) &= a_{21} y_1 + a_{22} y_2 + a_{23} y_3 + \cdots + a_{2n} y_n \\
&\ \vdots \\
y_n'(t) &= a_{n1} y_1 + a_{n2} y_2 + a_{n3} y_3 + \cdots + a_{nn} y_n,
\end{aligned}$$

introduce vector-matrix notation:

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

This gives $\mathbf{y}'(t) = A\mathbf{y}(t)$.

Transformations. 1. Transforming a single high-order equation into a system.
2. Transforming a system into a single high-order equation.

Solution via Eigenvectors. To solve $\mathbf{y}' = A\mathbf{y}$, guess $\mathbf{y} = e^{st}\mathbf{v}$ for some constant $s \in \mathbb{C}$ and constant vector $\mathbf{v}$. (Insist on $\mathbf{v} \neq \mathbf{0}$, since that choice leads to $\mathbf{y} = \mathbf{0}$, the trivial solution we can find by inspection.) This produces a solution iff

$$se^{st}\mathbf{v} = Ae^{st}\mathbf{v}, \quad \text{i.e.,} \quad A\mathbf{v} = s\mathbf{v}\ (\mathbf{v} \neq \mathbf{0}).$$

The latter equation is the definition of the statement, "$\mathbf{v}$ is an eigenvector for A with eigenvalue s." Every such pair gives one possible vector-valued solution, $e^{st}\mathbf{v}$; linearity produces the general solution

$$\mathbf{y}(t) = \sum_{k=1}^{n} c_k e^{s_k t} \mathbf{v}_k, \qquad k_1, \ldots, k_n \in \mathbb{R},$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent eigenvectors for A with corresponding eigenvalues $s_1, \ldots, s_n$. (Further discussion is required in cases of complex eigenvalues or an insufficient supply of eigenvectors.)

Related Skills.

- Solve initial-value problems based on $(*)$ when $y(0)$ and $y'(0)$ are known.
- Know Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and its applications.
- Know that for any constant $k > 0$, the limit as $x \to \infty$ gives

$$e^{-kx} \to 0, \quad xe^{-kx} \to 0, \quad x^2 e^{-kx} \to 0, \quad \ldots, \quad x^{316} e^{-kx} \to 0, \quad \cdots.$$

- Have some intuitive feeling for the profound difference between e^{at} with $a > 0$ and e^{at} with $a < 0$.
- Recall that when $a > 0$, $b > 0$, and $c > 0$, every solution for $(**)$ has $y(x) \to 0$ as $x \to \infty$.
- Recognize as particularly simple the case of $(**)$ where $b = 0$ with $ac > 0$ (simple harmonic oscillator).

B. Nonhomogeneous Linear ODE's with Constant Coefficients

Students are expected to be able to solve inhomogeneous problems

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = f(t) \tag{‡}$$

in cases where f is a sum of terms of the form

$$[\text{polynomial}] \times [\text{exponential}] \times [\text{sinusoid}]\,.$$

The Method of Undetermined Coefficients is standard here.

When solving an initial-value problem based on (‡), say

$$\begin{aligned} a\ddot{y} + b\dot{y} + cy &= f(t), \\ y(0) = y_0,\ \dot{y}(0) &= v_0, \end{aligned}$$

it's imperative to build the complete general solution of the full nonhomogeneous equation before applying the initial data. Doing these steps in the wrong order produces the wrong answer. To make sure everything is working, it's wise to check that whatever answer you write on the bottom line of a problem actually satisfies the stated initial conditions. (The sequence of steps in the outline below is chosen precisely to make correct sequencing automatic.)

It would be wonderful if the previous two paragraphs were all the instructor had to say in Math 257/316 to trigger a full and accurate recall by all students. Outside the "wonderful" case, the discussion below may be helpful.

Outline. Confronted with a problem involving an equation like (‡),

1. Guess one solution ["a particular solution"];
2. Build up a list of all solutions ["the general solution"];
3. Specialize/select as required [RTFQ; IC's; etc.].

1. Guessing a Particular Solution. This is like target shooting. The linear differential operator

$$L[y] = a\ddot{y} + b\dot{y} + cy$$

transforms an input function y into some new function. We want to know what to put in for y to match the "target", $f(x)$.

Example. Find one solution for $\ddot{y} + 4\dot{y} + 3y = 25e^{2t}$.

Solution. Guess $y = Ke^{2t}$. Plug in:

$$\begin{aligned} 4Ke^{2t} + 8Ke^{2t} + 3Ke^{2t} &= 25e^{2t} \\ 15Ke^{2t} &= 25e^{2t}. \end{aligned}$$

This works, with $K = 5/3$. One solution is $y_p = (5/3)e^{2t}$. ////

More generally, for the abstract operator L above,

$$y = e^{st} \implies L[y] = a(s^2 e^{st}) + b(se^{st}) + c(e^{st}) = (as^2 + bs + c)e^{st}. \qquad (*)$$

Put in an exponential function into L, and it generates the same exponential with a new constant multiplier. If the target is exponential, we should load the cannon with an exponential of the matching rate, but be prepared to adjust the constant.

Characteristic Polynomial. Recognize the factor $as^2 + bs + c$ in line $(*)$ as $p(s)$, the characteristic polynomial for L. For future use, note

$$p(s) = as^2 + bs + c \implies p'(s) = 2as + b \implies p''(s) = 2a. \qquad (**)$$

Exponential Shift. Imagine giving an input of form $y(t) = e^{st}u(t)$ to operator L. (Here s is a constant, possibly complex.) Use the product rule to find

$$\begin{aligned} \dot{y} &= se^{st}u + e^{st}\dot{u} = e^{st}\left(\dot{u} + su\right), \\ \ddot{y} &= s^2 e^{st}u + 2se^{st}\dot{u} + e^{st}\ddot{u} = e^{st}(\ddot{u} + 2s\dot{u} + s^2 u), \end{aligned}$$

then combine:

$$\begin{aligned} L[e^{st}u] &= ae^{st}\left(\ddot{u} + 2s\dot{u} + s^2 u\right) + be^{st}\left(\dot{u} + su\right) + ce^{st}u \\ &= e^{st}\left[a\ddot{u} + (2as + b)\dot{u} + (as^2 + bs + c)u\right] \end{aligned}$$

In view of the preview in line $(**)$ above, we have the wonderful *exponential shift formula*:

$$\boxed{L[e^{st}u] = e^{st}\left[\frac{p''(s)}{2!}\ddot{u} + \frac{p'(s)}{1!}\dot{u} + p(s)u\right].}$$

Back to target shooting: if the given $f(t)$ involves an exponential factor of the form e^{st}, use a matching exponential factor (i.e., the same numerical value for s) in the input $y(t) = e^{st}u(t)$. This will reduce your job to finding the factor $u(t)$.

Example (Same Problem, New Method). Find one solution for $\ddot{y} + 4\dot{y} + 3y = 25e^{2t}$.

Solution. Here $p(s) = s^2 + 4s + 3$, so $p'(s) = 2s + 4$ and $p''(s) = 2$. Since e^{2t} is a factor in the target function, choose $s = 2$ in the exponential shift formula. We will have

$$25e^{2t} = L[e^{2t}u(t)] = e^{2t}\left(\ddot{u} + 8\dot{u} + 15u\right)$$

if and only if $25 = \ddot{u} + 8\dot{u} + 15u$. One solution of this equation is the constant $u_p(t) = 25/15 = 5/3$. Therefore one solution of the original equation is

$$y_p(t) = e^{2t}u_p(t) = (5/3)e^{2t}.$$

This is the same result we found before, but it illustrates the new approach. (For problems this simple, the new approach is probably just a little slower than the direct method shown first. In contrast, the next example really shows the advantages of the exponential shift.) ////

Digression. For a general differential operator with constant coefficients like

$$L[y] = c_n y^{(n)} + c_{n-1} y^{(n-1)} + \ldots + c_2 \ddot{y} + c_1 \dot{y} + c_0 y,$$

the characteristic polynomial is

$$p(s) = c_n s^n + c_{n-1} s^{n-1} + \ldots + c_2 s^2 + c_1 s + c_0,$$

and the pattern suggested by the equation boxed above really works:

$$L[e^{st}u] = e^{st}\left[\frac{p^{(n)}(s)}{n!}u^{(n)} + \frac{p^{(n-1)}(s)}{(n-1)!}u^{(n-1)} + \ldots + \frac{p''(s)}{2!}\ddot{u} + \frac{p'(s)}{1!}\dot{u} + \frac{p(s)}{0!}u\right].$$

We seldom need this lovely fact for $n > 2$, but it also works for $n = 1$. That simple case is sometimes useful. ////

Example. Find one solution for $\ddot{y} + 4\dot{y} + 3y = -(2 + 8t)e^{-3t}$.

Solution. Here $p(s) = s^2 + 4s + 3$, so $p'(s) = 2s + 4$ and $p''(s) = 2$. If $y(t) = e^{-3t}u(t)$, exponential shift with $s = -3$ involves $p(-3) = 0$, $p'(-3) = -2$, $p''(-3) = 2$. So y provides a solution iff

$$e^{-3t}\left[\ddot{u} - 2\dot{u} + 0u\right] = -(2 + 8t)e^{-3t}, \quad \text{i.e.,} \quad \ddot{u} - 2\dot{u} = -2 - 8t.$$

To find one solution for u, try $u(t) = At^2 + Bt$: this requires

$$2A - 2(2At + B) = -2 - 8t.$$

Match coeffs of t: $A = 2$. Match constants: $B = 3$. So one solution for u is $u_p(t) = 2t^2 + 3t$, and it gives

$$y_p(t) = (2t^2 + 3t)e^{-3t}.$$

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Discussion. The reduced ODE for u could actually be used to find the general solution for y, not just one part of it. This is not usually a useful time-saver, but let's just watch it work in this example. The homogeneous counterpart of the u-equation above is $\ddot{u} - 2\dot{u} = 0$, and this is solved by both $u = e^{2t}$ and $u = 1$ (constant). Therefore the general solution for u is

$$u(t) = C_1 e^{2t} + C_2 + 2t^2 + 3t, \qquad C_1, C_2 \in \mathbb{R},$$

and this implies that the general solution for $y = e^{-3t}u$ must be

$$y(t) = C_1 e^{-t} + C_2 e^{-3t} + (2t^2 + 3t)e^{-3t}, \qquad C_1, C_2 \in \mathbb{R}.$$

2. Listing All Solutions. Back in (‡), suppose you have one solution, named $y_p(t)$. Look for others by splitting $y(t) = y_p(t) + w(t)$, where y_p is known and $w(t)$ is still to find. We need

$$f(t) = a\ddot{y} + b\dot{y} + cy = (a\ddot{y}_p + b\dot{y}_p + cy_p) + (a\ddot{w} + b\dot{w} + cw).$$

Here the choice of y_p is very helpful: our new unknown $w(t)$ must obey

$$a\ddot{w} + b\dot{w} + cw = 0.$$

We know how to generate a complete list of all solutions for this. Note that the characteristic polynomial for this new problem is the same $p(s)$ as for the original one. Skill with abstract thinking makes this clear: since the operator L is linear, we're just saying that if $L[y_p] = f$, then

$$f = L[y_p + w] \iff f = L[y_p] + L[w] \iff L[w] = 0.$$

Example. List all functions y satisfying $\quad \ddot{y} + 4\dot{y} + 3y = -(2 + 8t)e^{-3t}$.

Solution. We know one solution, $y_p = (2t^2 + 3t)e^{-3t}$. Solving

$$0 = p(s) = s^2 + 4s + 3 = (s + 3)(s + 1)$$

for $s = -1$, $s = -3$ reveals the general solution for the homogeneous problem:

$$L[w] = 0 \iff w(t) = Ae^{-t} + Be^{-3t}, \qquad A, B \in \mathbb{R}.$$

Therefore the general solution for the original nonhomogeneous equation is

$$y(t) = Ae^{-t} + Be^{-3t} + (2t^2 + 3t)e^{-3t}, \qquad A, B \in \mathbb{R}.$$

(Compare the item labelled "Discussion" above.) ////

3. Making a Selection. Auxiliary information about the desired function y should be applied only after an accurate list of candidates is available. That's why it's shown last in the suggested 3-step outline. Here's how the pieces already shown would fit into a full solution of an initial-value problem.

Example. Solve for $y(t)$:

$$\ddot{y} + 4\dot{y} + 3y = -(2 + 8t)e^{-3t}, \qquad y(0) = 1,\ \dot{y}(0) = 2.$$

Solution. First, find one solution of the given ODE, as shown earlier. (Recall $y_p(t) = (2t^2 + 3t)e^{-3t}$.) Second, augment that solution to produce the complete catalogue ("general solution") as above: recall

$$y(t) = Ae^{-t} + Be^{-3t} + (2t^2 + 3t)e^{-3t}, \qquad A, B \in \mathbb{R}.$$

Finally, third, enforce the extra conditions:

$$1 = y(0) = A + B, \qquad 2 = \dot{y}(0) = -A - 3B + 3.$$

These hold if and only if $A = 1$ and $B = 0$, so the stated problem has a unique solution, namely,

$$y(t) = e^{-t} + (2t^2 + 3t)e^{-3t}.$$

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C. Linear ODE's with Variable Coefficients–First Order

For the first-order linear ODE

$$y' + r(x)y = g(x), \tag{1}$$

everybody is expected to

- recognize the general solution form

  $$y = ky_1(x) + y_p(x), \qquad k \in \mathbb{R},$$

  where y_1 is any nontrivial solution of the homogeneous equation $y' + r(x)y = 0$ and $y_p(t)$ is any "particular solution" of (1),
- know how to find the general solution above,
- know how to find a specific solution when given a point (x_0, y_0) its graph must pass through (i.e., an "initial condition" like $y(x_0) = y_0$ for given numbers x_0 and y_0),
- know that the theory ensures existence and uniqueness for that IVP on any open interval containing x_0 throughout which functions $r(x)$ and $g(x)$ are continuous
- be on guard for possible bad behaviour at points where functions r and g are discontinuous.

D. Linear ODE's with Variable Coefficients–Euler Type

Observation. If $y = (x - x_0)^s$, then

$$\begin{aligned} y &= (x-x_0)^s, \\ y' &= s(x-x_0)^{s-1} = \frac{s}{(x-x_0)^1}\, y, \\ y'' &= s(s-1)(x-x_0)^{s-2} = \frac{s(s-1)}{(x-x_0)^2}\, y, \\ y''' &= s(s-1)(s-2)(x-x_0)^{s-3} = \frac{s(s-1)(s-2)}{(x-x_0)^3}\, y, \end{aligned}$$

so each of y, $(x-x_0)y'$, $(x-x_0)^2y''$, $(x-x_0)^3y'''$ is a constant multiple of the same function y. This is the key to handling equations of "Euler type", whose general form is

$$c_0y + c_1(x-x_0)y' + c_2(x-x_0)^2y'' + \cdots + c_n(x-x_0)^ny^{(n)} = 0.$$

[Here $c_0, \dots, c_n$ are real constants with $c_n \neq 0$.] The discussion is similar for all $n \geq 1$; we'll focus on the case $n = 2$, using simpler notation

$$a(x-x_0)^2y'' + b(x-x_0)y' + cy = 0. \tag{$*$}$$

"Euler-type" equations have the form

$$a(x-x_0)^2y'' + b(x-x_0)y' + cy = 0, \tag{$*$}$$

where a, b, c, x_0 are real constants. Assume $a \neq 0$ here. Then $(*)$ is equivalent to

$$y'' + \frac{b}{a(x-x_0)}y' + \frac{c}{a(x-x_0)^2}y = 0. \tag{$*'$}$$

Now $(*')$ is the standard form for theoretical developments, and here the coefficients are discontinuous at $x = x_0$. This is called a *singular point* of equation $(*)$, and we can expect solutions to exist separately in the intervals $(-\infty, x_0)$ and (x_0, ∞), but not on the whole real line. We'll work on the interval $(x_0, +\infty)$, leaving the other interval for home practice.

Solve $(*)$ in $(x_0, +\infty)$ by substituting $x - x_0 = e^t$ as shown below. (We did case $x_0 = 0$; adapt at home, please.) Or, just "guess" solution form $y = (x - x_0)^s$. Plug in, play, get $as^2 + (b-a)s + c = 0$. Three cases arise:

(i) Distinct real roots s_1, s_2, with $s_1 < s_2$: gen sol

$$y = k_1(x-x_0)^{s_1} + k_2(x-x_0)^{s_2},\ x > x_0, \qquad k_1, k_2 \in \mathbb{R}.$$

(ii) Repeated real root at s: gen sol

$$y = k_1(x-x_0)^{s} + k_2(x-x_0)^{s}\ln(x-x_0),\ x > x_0, \qquad k_1, k_2 \in \mathbb{R}.$$

(iii) Imaginary roots $s = \sigma \pm i\omega$ with $\omega > 0$: use $r = (x - x_0)$ in

$$r^s = r^\sigma x^{i\omega} = r^\sigma \left(e^{\ln r}\right)^{i\omega} = r^\sigma e^{i\omega \ln r} = r^\sigma \left[\cos(\omega \ln r) + i \sin(\omega \ln r)\right].$$

Separate real and imaginary parts to get the general solution for $x > x_0$:

$$y = (x - x_0)^\sigma \left[k_1 \cos(\omega \ln(x - x_0)) + k_2 \sin(\omega \ln(x - x_0))\right], \quad k_1, k_2 \in \mathbb{R}.$$

Summary Table (Case $x_0 = 0$)

	Constant-Coefficient Case	**Euler-type Equation**
Equation: "Guess": Quadratic for s:	$ay'' + by' + cy = 0, \quad x \in \mathbb{R}$ $y = e^{sx}$ $0 = as^2 + bs + c$	$ax^2y'' + bxy' + cy = 0, \quad x > 0$ $y = x^s$ $0 = as^2 + (b - a)s + c$
General solution cases: • $s_1 \neq s_2$ both real ... • $s_1 = s_2 = s$ ... • $s_1 = \sigma + i\omega$, $\omega \neq 0$ ...	 $y = Ae^{s_1 x} + Be^{s_2 x}$ $y = Ae^{sx} + Bxe^{sx}$ $y = e^{\sigma x}\left[A\cos(\omega x) + B\sin(\omega x)\right]$	 $y = Ax^{s_1} + Bx^{s_2}$ $y = Ax^s + B(\ln x)x^s$ $y = x^\sigma\left[A\cos(\omega \ln x) + B\sin(\omega \ln x)\right]$

Nonlinear Substitution. Assume $a \neq 0$ and $x_0 = 0$, and work on

$$ax^2y'' + bxy' + cy = 0. \tag{$\dagger$}$$

Here we work on the interval where $x > 0$, and substitute $x = e^t$: function y obeys (†) in the interval $(0, \infty)$ if and only if

$$a(e^t)^2 y''(e^t) + b(e^t)y'(e^t) + cy(e^t) = 0 \qquad \forall t \in \mathbb{R}.$$

Write $u(t) = y(e^t)$ and use the chain rule:

$$u'(t) = e^t y'(e^t), \qquad u''(t) = (e^t)^2 y''(e^t) + e^t y'(e^t).$$

Rearrange to eliminate y' and y'':

$$y'(e^t) = e^{-t}u'(t), \qquad y''(e^t) = \frac{u''(t) - e^t y'(e^t)}{e^{2t}} = e^{-2t}\left[u''(t) - u'(t)\right].$$

Back-substitute, watch wonderful cancellation occur, reduce (†) to

$$au''(t) + (b - a)u'(t) + cu(t) = 0, \qquad t \in \mathbb{R}. \tag{$\ddagger$}$$

Recognize "a problem we have solved before." Recall that $u(t) = y(x)$ when $x = e^t$, i.e., when $t = \ln x$. So typical solutions for (†) involve terms like these:

$$\begin{aligned} u(t) &= e^{st} &&\leftrightarrow \quad y(x) = u(\ln x) = x^s, \\ u(t) &= te^{st} &&\leftrightarrow \quad y(x) = u(\ln x) = x^s \ln(x), \\ u(t) &= e^{\sigma t}\cos(\omega t) &&\leftrightarrow \quad y(x) = u(\ln x) = x^\sigma \cos(\omega \ln x). \end{aligned}$$

But it's important to note where these s-values come from: the characteristic equation is not so easy to "read" from the coefficients in (†) ... it comes from (‡),

$$as^2 + (b - a)s + c = 0.$$

Sketches. Show some typical curves of $y = x^s$ for $s < 0$, $s = 0$, $s > 0$; then some curves for $y = x^s \ln x$; then typical $y = x^\sigma \cos(\omega \ln x)$.

Interval of Existence. Discuss situation for $x > x_0$ and $x < x_0$, and why it's unusual to have one function work in both regions. Explain why if $y = f(x - x_0)$ solves $(*)$ when $x > x_0$, then $u(t) = f(x_0 - t)$ solves $(*)$ when $t > x_0$.

Example. $x^2y'' - 2y = 0$. Find general solution in region $(0, +\infty)$. Find solution obeying $y(1) = 3$, $y'(1) = 0$. Find solution obeying $y(1) = 3$ with a finite limit as $x \to 0^+$. Find solution obeying $y(1) = 3$ with a finite limit as $x \to \infty$.

Answers. $y = k_1/x + k_2x^2$; $y' = -k_1/x^2 + 2k_2x$;

$$3 = k_1 + k_2,\ 0 = 2k_2 - k_1 \qquad \Longrightarrow\ k_1 = 2,\ k_2 = 1, \qquad y = 2/x + x^2.$$

$$\lim_{x\to 0^+} y(x) = \begin{cases} +\infty, & \text{if } k_1 > 0, \\ 0, & \text{if } k_1 = 0, \\ -\infty, & \text{if } k_1 < 0. \end{cases}$$

Hence $0 = k_1$, $3 = k_2$, giving $y(x) = 3x^2$.

$$\lim_{x\to \infty} y(x) = \begin{cases} +\infty, & \text{if } k_2 > 0, \\ 0, & \text{if } k_2 = 0, \\ -\infty, & \text{if } k_2 < 0. \end{cases}$$

Hence $0 = k_2$, $3 = k_1$, giving $y(x) = 3/x$. ////

Example. A certain nonzero function y, defined for $x > 2$, obeys $y(5) = 0$ and

$$(x-2)^2y'' + 5(x-2)y' + 8y = 0.$$

Find all zeros of y.

Solution. $y = (x-2)^{-2}\left[k_1\cos(2\ln(x-2)) + k_2\sin(2\ln(x-2))\right]$ for some k_1, k_2. Zero-spacing of $\phi(\theta) \stackrel{\text{def}}{=} k_1\cos(\theta) + k_2\sin(\theta)$ is π, so zeros will occur when

$$2\ln(x-2) - 2\ln(5-2) = k\pi,\ k \in \mathbb{Z},$$

i.e., when $x = 2 + \exp(\ln 3 + k\pi/2) = 2 + 3e^{k\pi/2}$, $k \in \mathbb{Z}$. ////

Nonhomogeneous Example by Substitution. Copy an old homework problem.