UBC Mathematics 257/316—Assignment 9
Due in class on Tuesday 22 July 2014

Themes: Sturm-Liouville Eigenvalue Problems in PDE; Nonhomogeneous ODE Review

1. Consider the eigenvalue problem

$$\begin{array}{lll} \text{(ODE)} & x^2y'' - 2xy' + 2y + \lambda x^2 y = 0, & 1 < x < 2 \\ \text{(BC)} & y(1) = 0, \qquad y(2) = 0. & \end{array}$$

(a) Find all the eigenvalues and eigenfunctions.

(b) Determine the orthogonality relation.

(c) Suppose the eigenvalue labels are chosen so that the full list of eigenvalues is $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with matching subscripts for the eigenfunctions. Determine the constants c_n compatible with the identity

$$x = \sum_{n=1}^{\infty} c_n y_n(x), \qquad 1 < x < 2.$$

Hint: To find exact solutions for (ODE), plug in $y(x) = xu(x)$ for some new unknown function u. This will produce an ODE for $u(x)$ that you know how to solve.

Credit: This is based on Trench, Section 13.2, problem 22.

2. Consider the following heat-flow problem:

$$\begin{array}{llll} \text{(PDE)} & u_t = (1-x^2)u_{xx} - 2xu_x, & 0 < x < 1, & t > 0, \\ \text{(BC)} & u(0,t) = 0, \qquad u(x,t) \text{ bounded for } 0 < x < 1, & & t > 0, \\ \text{(IC)} & u(x,0) = x^2, & 0 < x < 1. & \end{array}$$

(a) Show that $u(x,t) = e^{-\lambda t}X(x)$ provides a nontrivial solution for PDE/BC if and only if X solves a certain ODE eigenvalue problem for $0 \le x \le 1$.

(b) Verify that $X_1(x) = x$ and $X_2(x) = 5x^3 - 3x$ are eigenfunctions for the problem found in part (a). Find the corresponding eigenvalues, λ_1 and λ_2.

Note: *The eigenvalue problem in* (a) *is not fully compatible with the theory of regular Sturm-Liouville problems discussed in class, because the classroom discussion requires* $p(x) > 0$ *for each* x *in the closed interval* $[0,1]$, *and here we only have* $p(x) > 0$ *in the interval* $[0,1)$. *However, the main conclusions of that theory remain valid. You may use this fact in parts* (c)–(d).

(c) Suppose $X_1, X_2, \ldots,$ are eigenfunctions corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots,$ in the problem of part (a). Find the orthogonality relation satisfied by the family $\{X_1, X_2, \ldots\}$.

(d) Assuming that $u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$ solves the stated heat-flow problem, find the functions T_1 and T_2. (Use X_1 and X_2 from part (b).)

(continued)

3. Consider the following heat-flow problem:

$$\begin{array}{lll} \text{(PDE)} & u_t = u_{xx} - xu_x, & -1 < x < 1, \quad t > 0, \\ \text{(BC)} & u(-1,t) = 0, \qquad u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ \text{(IC)} & u(x,0) = 1 - x, & -1 < x < 1. \end{array}$$

(a) State a Sturm-Liouville eigenvalue problem whose eigenfunctions X appear in nontrivial separated-form solutions $u(x,t) = X(x)T(t)$ for PDE/BC.

(b) Find the orthogonality relation for the problem in (a).

(c) Verify that $X_1(x) = 1 - x^2$ is an eigenfunction for the problem in (a).

(d) Find the function $T_1(t)$ needed to write the solution to the heat-flow problem above as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = T_1(t)X_1(x) + T_2(t)X_2(x) + \cdots.$$

Here $X_1, X_2, \ldots$, are the eigenfunctions for the problem described in part (a), with $X_1(x) = 1 - x^2$ as in (b).

Note: Some nasty integrals come up. Instead of trying to solve them, just express them as suitable combinations of these constants:

$$\beta_n \overset{\text{def}}{=} \int_{-1}^{1} x^n e^{-x^2/2}\, dx, \qquad n = 0, 1, 2, \ldots.$$

(Note that $\beta_n = 0$ whenever n is odd.)

4. In the Trench textbook available online, read Section 5.4, and then solve the following problems. (Trench's approach to the Method of Undetermined Coefficients, a prerequisite topic, may be different from the one you saw in your earlier ODE course. You can use any correct method you like, but this one turns out to be very efficient.)

(i) Section 5.4, problem 6.

(ii) Section 5.4, problem 11.

(iii) Section 5.4, problem 24.

(iv) Section 5.4, problem 37(a).

(v) Section 5.4, problem 37(d).