UBC Mathematics 257/316—Assignment 3
Due in class on Tuesday 3 June 2014

Themes: Change of independent variable in ODE's, classifying singular points, Frobenius-style series solutions.

---

1. Use the change of variables $x = \sqrt{t}$ to find the general solution of

$$xy'' + (4x^2 - 1)y' + 20x^3 y = 0, \qquad x > 0.$$

2. (a) Consider the conditions

$$y''(t) + y(t) = 0, \quad t > 0; \qquad y(0) = 0,\ y'(0) \neq 0. \tag{$\dagger$}$$

(i) Prove that infinitely many different functions y satisfy ($\dagger$).

(ii) Prove that all y satisfying ($\dagger$) have the same zeros in $[0, \infty)$. List these zeros.

(b) Now let $\omega > 0$ be a constant, and consider the conditions

$$y''(x) + \omega^2 y(x) = 0, \quad x > 0; \qquad y(0) = 0,\ y'(0) \neq 0. \tag{$\ddagger$}$$

Use the substitution $x = at$ with a shrewd choice of constant $a > 0$ to reduce the following chores to jobs you have aready done in part (a):

(i) Prove that infinitely many different functions y satisfy ($\ddagger$).

(ii) Prove that all y satisfying ($\ddagger$) have the same zeros in $[0, \infty)$. List these zeros.

Remarks: Part (b) is pretty easy to solve directly, and that would be a reasonable way to check your answer. But the essential point here is to gain experience in using a stretching transformation like $x = at$ to relate a whole family of differential equations (as in ($\ddagger$)) to a single prototype with particularly convenient coefficients (as in ($\dagger$)). So following the instructions to "Use the substitution" is essential to get the learning benefit (and the marks) in this question.

3. Consider an initial-value problem of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \qquad y(0) = y_0,\ y'(0) = v_0, \tag{$*$}$$

where p, q, and f are continuous everywhere, while y_0 and v_0 are constants. Under these conditions, the requirements in ($*$) identify a unique function y.

Assume throughout this question that p is an *odd* function, and q is an *even* function.

(a) Prove: Whenever f is *even* and $v_0 = 0$, the solution y of ($*$) must be *even*.

(b) Prove: Whenever f is *odd* and $y_0 = 0$, the solution y of $(*)$ must be *odd*.

Clues: 1. A function h is *odd* when $h(-t) = -h(t)$ for all t, and *even* when $h(-t) = h(t)$ for all t. Only the zero-function is both; many functions are neither.

2. Suppose y satisfies $(*)$. Use this information to find an initial-value problem satisfied by the function $u(t) \stackrel{\text{def}}{=} y(-t)$. Recall that conditions $(*)$ identify a unique function.

4. Show that $x = 0$ is a regular singular point for the equation

$$9x^2y'' + (9x + 3x^3)y' - 4y = 0.$$

Find the recurrence relation and first three nonzero terms for each of two linearly independent series solutions about $x = 0$. What is the radius of convergence of the series in these solutions?

5. Find all singular points of each equation, and determine whether they are regular or irregular. At each regular singular point, find the indicial equation and the exponents of singularity.

(i) $(x + 2)^2(x - 1)y'' + 5(x - 1)y' - \pi(x + 2)y = 0$,

(ii) $2x^2y'' - 5(e^x - 1)y' + (e^{-x} \cos x)y = 0$.

6. Show that $x = 0$ is a regular singular point for the following equation:

$$x^2y'' + \left(x^2 + \frac{1}{4}\right) y = 0.$$

Find one series solution for this equation explicitly, and state the form (without finding all the constants) of a second series solution.

7. Consider the differential equation $(*)$: $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$.

(a) Show that $x = 0$ is a regular singular point for $(*)$.

(b) Find the exponents of singularity for $(*)$ about $x = 0$.

(c) Find a nonzero series-form solution of $(*)$ that is bounded on the interval $0 < x < 1$. [It suffices to find the recurrence relation and the first 4 nonzero terms.]