Supremum and Infimum
UBC M220 Lecture Notes by Philip D. Loewen

The Real Number System. Work hard to construct from the axioms a set \( \mathbb{R} \) with special elements \( \mathbb{O} \) and \( \mathbb{I} \), and a subset \( \mathbb{P} \subseteq \mathbb{R} \), and mappings \( A : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( M : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), for which defining the basic operations above in terms of
\[
x + y = A(x, y), \quad x \cdot y = M(x, y), \quad x > \mathbb{O} \iff x \in \mathbb{P}
\]
produces a consistent setup in which the familiar rules of arithmetic all work.

Trichotomy. For every real number \( x \), exactly one of the following is true:
\[
x < 0, \quad x = 0, \quad x > 0.
\]
By taking \( x = b - a \), we deduce that whenever \( a, b \in \mathbb{R} \), exactly one of the following is true:
\[
a < b, \quad a = b, \quad a > b.
\]
Given \( a, b \in \mathbb{R} \), it’s rather obvious that
\[
a > b \implies \exists \varepsilon > 0 : a \geq b + \varepsilon.
\]
(Indeed, if \( a > b \) then \( \varepsilon = a - b \) obeys the conclusion.) The contrapositive of this statement is logically equivalent, but occasionally useful:
\[
\left[ \forall \varepsilon > 0, \ a < b + \varepsilon \right] \implies a \leq b.
\]
It reveals that one way to prove “\( a \leq b \)” is to prove a collection of apparently easier inequalities involving a larger right-hand side.

Definition. Let \( S \subseteq \mathbb{R} \). To say, “\( S \) is bounded above,” means there exists \( b \in \mathbb{R} \) such that
\[
(*) \quad \forall s \in S, \ s \leq b.
\]
Any number \( b \) satisfying (*) is called “an upper bound for \( S \).”
Changing “\( \leq \)” to “\( > \)” in (*) produces a definition for the phrases “\( S \) is bounded below” and “\( b \) is a lower bound for \( S \).”
To say that \( S \) is bounded means that \( S \) is bounded above and \( S \) is bounded below.

Definition. Let \( S \subseteq \mathbb{R} \). The phrase, “\( \beta \) is a least upper bound for \( S \),” means two things:
(i) \( \forall s \in S, \ s \leq \beta \), i.e., \( \beta \) is an upper bound for \( S \), i.e.,
\[
(*) \quad \forall s \in S, \ s \leq \beta.
\]
(ii) Every real number less than \( \beta \) is not an upper bound for \( S \). Express this second condition as

\[(**)
\forall \varepsilon > 0, \exists s \in S : \beta - \varepsilon < s .\]

**Notation.** A given set \( S \) can have at most one least upper bound (LUB).

[Pf: Suppose \( \beta_0 \) is a least upper bound for \( S \). Any real \( \beta > \beta_0 \) breaks (**)—use \( \varepsilon = \beta - \beta_0 \) and recall (*). Any real \( \beta < \beta_0 \) breaks (*)—use \( \varepsilon = \beta_0 - \beta \) and recall (**).]

If \( S \) has a least upper bound, it is a unique element of \( \mathbb{R} \) denoted

\[\sup S \quad \text{(Latin supremum).}\]

A symmetric development leads to the concepts of sets bounded below, greatest lower bounds, and the Latin notation

\[\inf S \quad \text{(Latin infimum).}\]

**The Least Upper Bound Property.** The hard work in the axiomatic construction of the real number system is in arranging the following fundamental property:

For each nonempty subset \( S \) of \( \mathbb{R} \), if \( S \) has an upper bound, then there exists a unique real number \( \beta \) such that \( \beta = \sup S \).

[Alternate terminology: \((\mathbb{R}, \leq)\) is order-complete.]

**Proposition.** \( \mathbb{R} \) has the greatest lower bound property, i.e.,

For each nonempty subset \( S \) of \( \mathbb{R} \), if \( S \) has a lower bound, then there exists a unique real number \( \alpha \) such that \( \alpha = \inf S \).

**Proof.** Consider any subset \( S \) of \( \mathbb{R} \), assuming \( S \neq \emptyset \) and \( S \) has a lower bound. Define

\[ L = \{ x \in \mathbb{R} : x \text{ is a lower bound for } S \} .\]

Note \( L \neq \emptyset \) by hypothesis; the definition of \( L \) gives

\[ \forall x \in L, \forall s \in S, x \leq s .\quad (*)\]

This is equivalent to

\[ \forall s \in S, \forall x \in L, x \leq s .\quad (**)\]

The latter form shows that any \( s \) in \( S \) provides an upper bound for the set \( L \); since \( S \neq \emptyset \) by hypothesis, it follows that \( L \) has an upper bound. By the LUB Property, \( \alpha = \sup(L) \) is a well-defined real number. We’ll show that \( \alpha \) is the desired greatest lower bound for \( S \).

First, \( \alpha \) is a lower bound for \( S \). For otherwise, there would be some \( s \) in \( S \) with \( s < \alpha \); this \( s \) would be an upper bound for \( L \) [see (***) above], contradicting the definition of \( \alpha \) as the least upper bound for \( L \).

Second, \( \alpha \) is the greatest lower bound for \( S \). Indeed, consider any \( \gamma > \alpha \): by construction, \( \alpha \) is an upper bound for \( L \), so \( \alpha \geq x \forall x \in L \). Consequently \( \gamma \not\in L \), i.e., \( \gamma \) is not a lower bound for \( S \).
Relevance: Existence Theorems. The central feature of the LUB Property is the statement that there exists a real number (the supremum) with certain properties. Thus it provides the foundation for all interesting theorems involving existence of certain mathematical objects. (E.g., the IVT from Calculus: If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there exists a real number $x \in (a, b)$ obeying $f(x) = 0$.) The foundation can serve in two ways:

(i) Directly: The desired number may be defined as a supremum. E.g., to prove $\exists x \in \mathbb{R} : x^2 = 2$, we appeal to order-completeness to assert that $\alpha = \sup \{q \in \mathbb{Q} : q^2 < 2\}$ is a well-defined real number, then use (*)--(**) to show that $\alpha^2 = 2$.

(ii) Indirectly: The definition of “sup” ensures the existence of near-maximal elements. For example, suppose $X$ is some nonempty set, and $f: X \rightarrow \mathbb{R}$ is a function whose range $f(X)$ is bounded above. Then $\beta = \sup f(X) = \sup \{f(x) : x \in X\}$ is a well-defined real number. We cannot assert $\beta \in f(X)$ without knowing more about $f$ and $X$, but we can argue as follows: for each $n \in \mathbb{N}$, $\beta - 1/n$ is less than the least upper bound for $f(X)$, so it is not an upper bound. This means that some element of $f(X)$—call it $y_n$—obeys $y_n > \beta - 1/n$. Choose some element of $X$, and name it $x_n$, satisfying $y_n = f(x_n)$. This procedure creates a sequence $x_1, x_2, \ldots$ in $X$ with the useful property

$$\forall n \in \mathbb{N}, \quad \beta - \frac{1}{n} < f(x_n) \leq \beta.$$  

[Such a sequence is called a “maximizing sequence” for $f$.]

Theorem (Archimedes). In $\mathbb{R}$, the set $\mathbb{N}$ has no upper bound. That is,

$$\forall r \in \mathbb{R}, \exists n \in \mathbb{N} : n > r.$$  

Proof. (By contradiction.) Suppose, on the contrary, that $\mathbb{N}$ has an upper bound. Evidently $\mathbb{N} \neq \emptyset$, so by LUB property, $\beta = \sup \mathbb{N}$ exists in $\mathbb{R}$. Consider $\gamma \overset{\text{def}}{=} \beta - 1$: (**) gives some $n$ in $\mathbb{N}$ such that $\gamma < n$. Hence $\beta - 1 < n$, i.e., $\beta < n + 1$. Since $n + 1 \in \mathbb{N}$, this contradicts property (*) for $\beta$. // // //

Corollaries. (a) For any fixed $\varepsilon > 0$, some $n \in \mathbb{N}$ obeys $1/n < \varepsilon$.

(b) Whenever $x, y \in \mathbb{R}$ obey $y - x > 1$, we have $(x, y) \cap \mathbb{Z} \neq \emptyset$.

(c) For any $a, b \in \mathbb{R}$ with $a < b$, we have both $(a, b) \cap \mathbb{Q} \neq \emptyset$ and $(a, b) \setminus \mathbb{Q} \neq \emptyset$.

Proof. (a) Apply Archimedes to $r = 1/\varepsilon$ to produce $n \in \mathbb{N}$ s.t. $n > 1/\varepsilon$, i.e., $1/n < \varepsilon$.

(b) Let $S = \{n \in \mathbb{Z} : n \geq y\}$. By Archimedes, $S \neq \emptyset$; by Fact 1, $\hat{n} = \min(S)$ exists. Let’s show $z = \hat{n} - 1 \in (x, y)$:

(i) $z < y$: indeed, $z \geq y$ would imply $z \in S$, contradicting minimality of $\hat{n}$.  

(ii) \( z > x \): indeed, \( \hat{n} \in S \implies \hat{n} \geq y > x + 1 \implies \hat{n} - 1 > x \).

(c) Given \( a < b \), apply (a) to get \( n \in \mathbb{N} \) such that \( 1/n < b - a \). Then \( nb - na > 1 \), so (b) applies to \( x = na, y = nb \): some \( m \in \mathbb{Z} \) obeys \( na < m < nb \), i.e., \( a < \frac{m}{n} < b \). Thus \( \frac{m}{n} \in (a, b) \cap \mathbb{Q} \).

Likewise, if \( a < b \) then \( \frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}} \) so some \( q \in \mathbb{Q} \) obeys \( \frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}} \). It follows that \( q\sqrt{2} \in (a, b) \setminus \mathbb{Q} \).

\[ \text{Home Practice.} \] Let \( S = \{1/n : n \in \mathbb{N}\} \). Use Archimedes to justify “\( \inf(S) = 0 \)” [check \((*)-(**)\)]. Note \( 0 \not\in S \). [Write \( \beta = \min S \) when both \( \beta = \inf(S) \) and \( \beta \in S \), saying “the infimum is \textbf{attained}”. So for this \( S \), \( \inf(S) = 0 \) but \( \min(S) \) does not exist. Similarly, “\( \max \)” means “attained supremum.”]

\textbf{Easy Example (Sketch Steps Only).} Given subsets \( A \) and \( B \) of \( \mathbb{R} \) such that \( A \neq \emptyset \), \( B \) is bounded above, and \( A \subseteq B \), show that \( \sup A \leq \sup B \).

(i) Show \( \sup B \) exists. [ETS \( B \neq \emptyset \), \( B \) has upper bound.]

Proof: Since \( A \neq \emptyset \), some \( x \) obeys \( x \in A \). Since \( A \subseteq B \), this same \( x \) obeys \( x \in B \). Therefore \( B \neq \emptyset \). Now \( B \) is bounded above by assumption, so \( \sup B \) exists. Call it \( \beta \).

(ii) Show \( \sup A \) exists. [ETS \( A \neq \emptyset \), \( A \) has upper bound.]

Proof: Since \( B \) is bounded above, there exists some \( M \) satisfying \( y \leq M \) for all \( y \in B \). Since \( A \subseteq B \), every \( x \) in \( A \) obeys \( x \in B \), and hence \( x \leq M \). Thus \( M \) is an upper bound for \( A \), while \( A \neq \emptyset \) is given. It follows that \( \alpha \overset{\text{def}}{=} \sup A \) exists in \( \mathbb{R} \).

(iii) Show \( \sup A \leq \sup B \). [Define \( \beta = \sup B \). Assume \( \beta < \sup A \), get contradiction.]

The argument in (ii) shows that any upper bound for \( B \) is an upper bound for \( A \). In particular, \( \beta = \sup B \) must be an upper bound for \( A \). To show \( \beta \geq \alpha \), imagine the alternative: If \( \beta < \alpha \), then \( \beta \) is not an upper bound for \( A \) (by definition of \( \alpha = \sup A \)), a contradiction. We must have \( \beta \geq \alpha \).

(iv) T/F? Strict inclusion \( A \subseteq B \), \( A \neq B \), implies strict inequality \( \sup A < \sup B \).

False: Consider, e.g., \( A = (0, 1) \) and \( B = [0, 1] \). Here \( A \subseteq B \), \( A \neq B \), yet \( \sup(A) = 1 = \sup(B) \).

\[ \text{////} \]

\textbf{Monotone Sequences}

\textbf{Definition.} Let a real-valued sequence \( (a_n)_{n \in \mathbb{N}} \) be given.

(a) Call \( (a_n) \) nondecreasing when \( n < m \implies a_n \leq a_m \);

(b) Call \( (b_n) \) nonincreasing when \( n < m \implies a_n \geq a_m \).

Call \( (a_n) \) \textit{monotone} when it is either nondecreasing or nonincreasing.
Theorem. Let sequence \((a_n)_{n \in \mathbb{N}}\) be monotone.

(a) If \((a_n)\) is nondecreasing and bounded above, then \(a_n \to \beta\) as \(n \to \infty\), where 
\[\beta = \sup \{a_n\}.\]

(b) If \((a_n)\) is nonincreasing and bounded below, then \(a_n \to \alpha\) as \(n \to \infty\), where 
\[\alpha = \inf \{a_n\}.\]

Proof. (a) Suppose the set of sequence entries \(A = \{a_n : n \in \mathbb{N}\}\) has an upper bound. Then \(\beta \overset{\text{def}}{=} \sup \{a_n : n \in \mathbb{N}\}\) is a unique real number, and it is an upper bound for \(A\). So 
\[\forall n \in \mathbb{N}, \quad a_n \leq \beta.\]

But \(\beta\) is the least upper bound for \(A\), so given any \(\varepsilon > 0\), the number \(\beta - \varepsilon\) is not an upper bound for \(A\). That is, some element of \(A\) must be larger than \(\beta - \varepsilon\). But all the elements of \(A\) are sequence entries, so there must be some positive integer \(N\) such that \(a_N > \beta - \varepsilon\). Now since the sequence is nondecreasing, every integer \(n > N\) will have 
\[\beta - \varepsilon < a_N \leq a_n \leq \beta.\]

This certainly implies 
\[\forall n > N, \; |a_n - \beta| < \varepsilon.\]

(b) Similar. (Try it!) \(////\)

Application. Classic problem genre: Prove that a sequence converges without even guessing its limit, by showing that the sequence is monotonic and bounded. Prove those properties by induction.

Example. Let 
\[x_1 = \sqrt{2} \text{ and } x_{n+1} = \sqrt{2 + x_n} \text{ for each } n \in \mathbb{N}.\]
Prove that \((x_n)_{n \in \mathbb{N}}\) converges, and find the limit.

Solution. If the sequence converges to \(\hat{x}\), then sending \(n \to \infty\) on both sides of the iteration equation \(x_{n+1} = \sqrt{2 + x_n}\) would give \(\hat{x} = \sqrt{2 + \hat{x}}\). Thus \(\hat{x} > \sqrt{2}\) and 
\[0 = \hat{x}^2 - \hat{x} - 2 = (\hat{x} - 2)(\hat{x} + 1),\]
which would give \(\hat{x} = 2\). This is useful preliminary information that sets up a careful appeal to the monotone sequence theorem.

Let’s show that for each \(n \in \mathbb{N}\), the following statement is true:
\[P(n) : \quad x_n \leq x_{n+1} \leq 2.\]

Mathematical induction is effective here.

Base Case: Statement \(P(1)\) says \(\sqrt{2} \leq \sqrt{2 + \sqrt{2}} \leq 2\). This is true.

Induction Step: Suppose \(n \in \mathbb{N}\) is an integer for which statement \(P(n)\) is true. We would like to deduce the following.
\[P(n+1) : \quad x_{n+1} \leq x_{n+2} \leq 2.\]
To get this, add 2 to each entry in statement $P(n)$. This gives

$$2 + x_n \leq 2 + x_{n+1} \leq 4.$$ 

These numbers are all positive, so their square roots must come in the same order:

$$\sqrt{2 + x_n} \leq \sqrt{2 + x_{n+1}} \leq \sqrt{4}.$$ 

The iteration formula allows us to rearrange this as

$$x_{n+1} \leq x_{n+2} \leq 2,$$

which is precisely the outcome we seek.

By induction, statement $P(n)$ is true for each $n \in \mathbb{N}$. Thus the sequence $(x_n)$ is nondecreasing and bounded above, so it must converge. The value of the limit must be 2, for reasons given earlier.