Supremum and Infimum

UBC M220 Lecture Notes by Philip D. Loewen

The Real Number System. Work hard to construct from the axioms a set $\mathbb{R}$ with special elements $\mathbb{O}$ and $\mathbb{I}$, and a subset $\mathbb{P} \subseteq \mathbb{R}$, and mappings $A\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $M\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for which defining the basic operations above in terms of

$$x + y = A(x, y), \qquad x \cdot y = M(x, y), \qquad x > \mathbb{O} \Leftrightarrow x \in \mathbb{P}$$

produces a consistent setup in which the familiar rules of arithmetic all work.

Trichotomy. For every real number x, exactly one of the following is true:

$$x < 0, \qquad x = 0, \qquad x > 0.$$

By taking $x = b - a$, we deduce that whenever $a, b \in \mathbb{R}$, exactly one of the following is true:

$$a < b, \qquad a = b, \qquad a > b.$$

Given $a, b \in \mathbb{R}$, it's rather obvious that

$$a > b \implies \exists \varepsilon > 0 \,:\, a \geq b + \varepsilon.$$

(Indeed, if $a > b$ then $\varepsilon = a - b$ obeys the conclusion.) The contrapositive of this statement is logically equivalent, but occasionally useful:

$$\boxed{\Big[\forall \varepsilon > 0,\ a < b + \varepsilon\Big] \implies a \leq b.}$$

It reveals that one way to prove "$a \leq b$" is to prove a collection of apparently easier inequalities involving a larger right-hand side.

Definition. Let $S \subseteq \mathbb{R}$. To say, "S is bounded above," means there exists $b \in \mathbb{R}$ such that

$$(*) \qquad \forall s \in S,\ s \leq b.$$

Any number b satisfying $(*)$ is called "an upper bound for S."
Changing "$\leq$" to "$\geq$" in $(*)$ produces a definition for the phrases "S is bounded below" and "b is a lower bound for S".
To say that S is bounded means that S is bounded above and S is bounded below.

Definition. Let $S \subseteq \mathbb{R}$. The phrase, "β is a least upper bound for S," means two things:

(i) $\forall s \in S,\ s \leq \beta$, i.e., β is an upper bound for S, i.e.,

$$(*) \qquad \forall s \in S,\ s \leq \beta.$$

(ii) Every real number less than β is *not* an upper bound for S. Express this second condition as

$$(**) \qquad \forall \varepsilon > 0,\ \exists s \in S\ :\ \beta - \varepsilon < s.$$

Notation. A given set S can have at most one least upper bound (LUB).
[Pf: Suppose β_0 is a least upper bound for S. Any real $\beta > \beta_0$ breaks $(**)$—use $\varepsilon = \beta - \beta_0$ and recall $(*)$. Any real $\beta < \beta_0$ breaks $(*)$—use $\varepsilon = \beta_0 - \beta$ and recall $(**)$.]

If S has a least upper bound, it is a unique element of $\mathbb{R}$ denoted

$$\sup S \qquad \text{(Latin } \textit{supremum}\text{)}.$$

A symmetric development leads to the concepts of sets bounded below, greatest lower bounds, and the Latin notation

$$\inf S \qquad \text{(Latin } \textit{infimum}\text{)}.$$

The Least Upper Bound Property. The hard work in the axiomatic construction of the real number system is in arranging the following fundamental property:

> For each nonempty subset S of $\mathbb{R}$, if S has an upper bound, then there exists a unique real number β such that $\beta = \sup S$.

[Alternate terminology: $(\mathbb{R}, \leq)$ is **order-complete**.]

Proposition. $\mathbb{R}$ *has the greatest lower bound property, i.e.,*

> *For each nonempty subset S of $\mathbb{R}$, if S has a lower bound, then there exists a unique real number α such that $\alpha = \inf S$.*

Proof. Consider any subset S of $\mathbb{R}$, assuming $S \neq \emptyset$ and S has a lower bound. Define

$$L = \{x \in \mathbb{R}\ :\ x \text{ is a lower bound for } S\}.$$

Note $L \neq \emptyset$ by hypothesis; the definition of L gives

$$\forall x \in L,\ \forall s \in S,\ x \leq s. \tag{$*$}$$

This is equivalent to

$$\forall s \in S,\ \forall x \in L,\ x \leq s. \tag{$**$}$$

The latter form shows that any s in S provides an upper bound for the set L; since $S \neq \emptyset$ by hypothesis, it follows that L has an upper bound. By the LUB Property, $\alpha = \sup(L)$ is a well-defined real number. We'll show that α is the desired greatest lower bound for S.

First, α is *a* lower bound for S. For otherwise, there would be some s in S with $s < \alpha$; this s would be an upper bound for L [see $(**)$ above], contradicting the definition of α as the *least* upper bound for L.

Second, α is the *greatest* lower bound for S. Indeed, consider any $\gamma > \alpha$: by construction, α is an upper bound for L, so $\alpha \geq x\ \forall x \in L$. Consequently $\gamma \notin L$, i.e., γ is not a lower bound for S. ////

Relevance: Existence Theorems. The central feature of the LUB Property is the statement that *there exists a real number* (the supremum) with certain properties. Thus it provides the foundation for all interesting theorems involving existence of certain mathematical objects. (E.g., the IVT from Calculus: If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then *there exists a real number* $x \in (a,b)$ obeying $f(x) = 0$.) The foundation can serve in two ways:

(i) Directly: The desired number may be defined as a supremum. E.g., to prove "$\exists x \in \mathbb{R} : x^2 = 2$", we appeal to order-completeness to assert that

$$\alpha = \sup\left\{q \in \mathbb{Q} : q^2 < 2\right\}$$

is a well-defined real number, then use $(*)$–$(**)$ to show that $\alpha^2 = 2$.

(ii) Indirectly: The definition of "sup" ensures the existence of near-maximal elements. For example, suppose X is some nonempty set, and $f\colon X \to \mathbb{R}$ is a function whose range $f(X)$ is bounded above. Then $\beta = \sup f(X) = \sup\{f(x) : x \in X\}$ is a well-defined real number. We cannot assert $\beta \in f(X)$ without knowing more about f and X, but we can argue as follows: for each $n \in \mathbb{N}$, $\beta - 1/n$ is less than the least upper bound for $f(X)$, so it is not an upper bound. This means that some element of $f(X)$—call it y_n—obeys $y_n > \beta - 1/n$. Choose some element of X, and name it x_n, satisfying $y_n = f(x_n)$. This procedure creates a sequence $x_1, x_2, \ldots$ in X with the useful property

$$\forall n \in \mathbb{N}, \quad \beta - \frac{1}{n} < f(x_n) \le \beta.$$

[Such a sequence is called a "maximizing sequence" for f.]

Theorem (Archimedes). *In $\mathbb{R}$, the set $\mathbb{N}$ has no upper bound. That is,*

$$\forall r \in \mathbb{R},\ \exists n \in \mathbb{N} : n > r.$$

Proof. (By contradiction.) Suppose, on the contrary, that $\mathbb{N}$ has an upper bound. Evidently $\mathbb{N} \neq \emptyset$, so by LUB property, $\beta = \sup \mathbb{N}$ exists in $\mathbb{R}$. Consider $\gamma \stackrel{\text{def}}{=} \beta - 1$: $(**)$ gives some n in $\mathbb{N}$ such that $\gamma < n$. Hence $\beta - 1 < n$, i.e., $\beta < n + 1$. Since $n + 1 \in \mathbb{N}$, this contradicts property $(*)$ for β. ////

Corollaries. (a) *For any fixed $\varepsilon > 0$, some $n \in \mathbb{N}$ obeys $1/n < \varepsilon$.*

(b) *Whenever $x, y \in \mathbb{R}$ obey $y - x > 1$, we have $(x,y) \cap \mathbb{Z} \neq \emptyset$.*

(c) *For any $a, b \in \mathbb{R}$ with $a < b$, we have both $(a,b) \cap \mathbb{Q} \neq \emptyset$ and $(a,b) \setminus \mathbb{Q} \neq \emptyset$.*

Proof. (a) Apply Archimedes to $r = 1/\varepsilon$ to produce $n \in \mathbb{N}$ s.t. $n > 1/\varepsilon$, i.e., $1/n < \varepsilon$.

(b) Let $S = \{n \in \mathbb{Z} : n \ge y\}$. By Archimedes, $S \neq \emptyset$; by Fact 1, $\widehat{n} = \min(S)$ exists. Let's show $z = \widehat{n} - 1 \in (x,y)$:

(i) $z < y$: indeed, $z \ge y$ would imply $z \in S$, contradicting minimality of $\widehat{n}$.

(ii) $z > x$: indeed, $\widehat{n} \in S \implies \widehat{n} \geq y > x + 1 \implies \widehat{n} - 1 > x$.

(c) Given $a < b$, apply (a) to get $n \in \mathbb{N}$ such that $1/n < b - a$. Then $nb - na > 1$, so (b) applies to $x = na$, $y = nb$: some $m \in \mathbb{Z}$ obeys $na < m < nb$, i.e., $a < \dfrac{m}{n} < b$. Thus $\dfrac{m}{n} \in (a, b) \cap \mathbb{Q}$.

Likewise, if $a < b$ then $\dfrac{a}{\sqrt{2}} < \dfrac{b}{\sqrt{2}}$ so some $q \in \mathbb{Q}$ obeys $\dfrac{a}{\sqrt{2}} < q < \dfrac{b}{\sqrt{2}}$. It follows that $q\sqrt{2} \in (a, b) \setminus \mathbb{Q}$. ////

Home Practice. Let $S = \{1/n : n \in \mathbb{N}\}$. Use Archimedes to justify "$\inf(S) = 0$" [check $(*)$–$(**)$]. Note $0 \notin S$. [Write $\beta = \min S$ when both $\beta = \inf(S)$ and $\beta \in S$, saying "the infimum is **attained**". So for this S, $\inf(S) = 0$ but $\min(S)$ does not exist. Similarly, "max" means "attained supremum."]

Easy Example (Sketch Steps Only). *Given subsets A and B of $\mathbb{R}$ such that $A \neq \emptyset$, B is bounded above, and $A \subseteq B$, show that* $\sup A \leq \sup B$.

(i) Show $\sup B$ exists. [ETS $B \neq \emptyset$, B has upper bound.]

Proof: Since $A \neq \emptyset$, some x obeys $x \in A$. Since $A \subseteq B$, this same x obeys $x \in B$. Therefore $B \neq \emptyset$. Now B is bounded above by assumption, so $\sup B$ exists. Call it β.

(ii) Show $\sup A$ exists. [ETS $A \neq \emptyset$, A has upper bound.]

Proof: Since B is bounded above, there exists some M satisfying $y \leq M$ for all $y \in B$. Since $A \subseteq B$, every x in A obeys $x \in B$, and hence $x \leq M$. Thus M is an upper bound for A, while $A \neq \emptyset$ is given. It follows that $\alpha \stackrel{\text{def}}{=} \sup A$ exists in $\mathbb{R}$.

(iii) Show $\sup A \leq \sup B$. [Define $\beta = \sup B$. Assume $\beta < \sup A$, get contradiction.]

The argument in (ii) shows that any upper bound for B is an upper bound for A. In particular, $\beta = \sup B$ must be an upper bound for A. To show $\beta \geq \alpha$, imagine the alternative: If $\beta < \alpha$, then β is not an upper bound for A (by definition of $\alpha = \sup A$), a contradiction. We must have $\beta \geq \alpha$.

(iv) T/F? Strict inclusion $A \subseteq B$, $A \neq B$, implies strict inequality $\sup A < \sup B$.

False: Consider, e.g., $A = (0, 1)$ and $B = [0, 1]$. Here $A \subseteq B$, $A \neq B$, yet $\sup(A) = 1 = \sup(B)$. ////

Monotone Sequences

Definition. Let a real-valued sequence $(a_n)_{n \in \mathbb{N}}$ be given.

(a) Call (a_n) nondecreasing when $n < m \implies a_n \leq a_m$;

(b) Call (b_n) nonincreasing when $n < m \implies a_n \geq a_m$.

Call (a_n) *monotone* when it is either nondecreasing or nonincreasing.

Theorem. *Let sequence* $(a_n)_{n\in\mathbb{N}}$ *be monotone.*

(a) *If* (a_n) *is nondecreasing and bounded above, then* $a_n \to \beta$ *as* $n \to \infty$*, where* $\beta = \sup\{a_n\}$.

(b) *If* (a_n) *is nonincreasing and bounded below, then* $a_n \to \alpha$ *as* $n \to \infty$*, where* $\alpha = \inf\{a_n\}$.

Proof. (a) Suppose the set of sequence entries $A = \{a_n : n \in \mathbb{N}\}$ has an upper bound. Then $\beta \stackrel{\text{def}}{=} \sup\{a_n : n \in \mathbb{N}\}$ is a unique real number, and it is an upper bound for A. So

$$\forall n \in \mathbb{N}, \quad a_n \le \beta.$$

But β is the *least* upper bound for A, so given any $\varepsilon > 0$, the number $\beta - \varepsilon$ is not an upper bound for A. That is, some element of A must be larger than $\beta - \varepsilon$. But all the elements of A are sequence entries, so there must be some positive integer N such that $a_N > \beta - \varepsilon$. Now since the sequence is nondecreasing, every integer $n > N$ will have

$$\beta - \varepsilon < a_N \le a_n \le \beta.$$

This certainly implies

$$\forall n > N, \ |a_n - \beta| < \varepsilon.$$

(b) Similar. (Try it!) ////

Application. Classic problem genre: Prove that a sequence converges without even guessing its limit, by showing that the sequence is monotonic and bounded. Prove those properties by induction.

Example. *Let* $x_1 = \sqrt{2}$ *and* $x_{n+1} = \sqrt{2 + x_n}$ *for each* $n \in \mathbb{N}$*. Prove that* $(x_n)_{n\in\mathbb{N}}$ *converges, and find the limit.*

Solution. If the sequence converges to $\widehat{x}$, then sending $n \to \infty$ on both sides of the iteration equation $x_{n+1} = \sqrt{2 + x_n}$ would give $\widehat{x} = \sqrt{2 + \widehat{x}}$. Thus $\widehat{x} > \sqrt{2}$ and

$$0 = \widehat{x}^2 - \widehat{x} - 2 = (\widehat{x} - 2)(\widehat{x} + 1),$$

which would give $\widehat{x} = 2$. This is useful preliminary information that sets up a careful appeal to the monotone sequence theorem.

Let's show that for each $n \in \mathbb{N}$, the following statement is true:

$$P(n): \qquad x_n \le x_{n+1} \le 2.$$

Mathematical induction is effective here.

Base Case: Statement $P(1)$ says $\sqrt{2} \le \sqrt{2 + \sqrt{2}} \le 2$. This is true.

Induction Step: Suppose $n \in \mathbb{N}$ is an integer for which statement $P(n)$ is true. We would like to deduce the following.

$$P(n+1): \qquad x_{n+1} \le x_{n+2} \le 2.$$

To get this, add 2 to each entry in statement $P(n)$. This gives

$$2 + x_n \leq 2 + x_{n+1} \leq 4.$$

These numbers are all positive, so their square roots must come in the same order:

$$\sqrt{2 + x_n} \leq \sqrt{2 + x_{n+1}} \leq \sqrt{4}.$$

The iteration formula allows us to rearrange this as

$$x_{n+1} \leq x_{n+2} \leq 2,$$

which is precisely the outcome we seek.

By induction, statement $P(n)$ is true for each $n \in \mathbb{N}$. Thus the sequence (x_n) is nondecreasing and bounded above, so it must converge. The value of the limit must be 2, for reasons given earlier. ////