Sample Calculus Exam #1

Instructions Any calculator that is allowed in the Principles of Mathematics 12 provincial examination may be used, but will not be necessary. Answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.

Questions 1–8 are "short answer" questions, and only the final result will be marked; in particular, you need not show your work. In questions 9–14, you need to show your work in detail to receive full credit.

1. [8] a) Let $f(x) = \tan x$. Find $f''(x)$, the second derivative of $f(x)$.

b) Let $f(x) = x \arctan x - (1/2)\ln(1 + x^2)$. Find $f'(x)$ and simplify.

2. [4] Find $g'(3)$ if $g(x) = x2^{h(x)}$ where $h(3) = -2$ and $h'(3) = 5$.

3. [4] Find the slope of the tangent line to the curve $y + x \ln y - 2x = 0$ at the point $(1/2, 1)$.

4. [6] Let $f(x) = \dfrac{2x}{x^2 + 3}$.

a) Write an equation of the tangent line to the curve $y = f(x)$ at $x = 1$.

b) Use linear approximation to give an approximate value for $f(1.2)$.

5. [6] A particle moves along the x-axis so that its position at time t is given by $x = t^3 - 4t^2 + 1$.

a) At $t = 2$, what is the particle's speed?

b) At $t = 2$, in what direction is the particle moving?

c) At $t = 2$, is the particle's speed increasing or decreasing?

6. [5] Let $g(x) = \arcsin(\cos x)$.

a) Calculate and simplify the derivative $g'(x)$.

b) At what points does $g'(x)$ fail to exist?

7. [5] Suppose that $f'(2) = 3$. Find $\displaystyle\lim_{x \to 2} \frac{f(x) - f(2)}{\sqrt{x} - \sqrt{2}}$.

8. [6] Suppose that the function y satisfies the differential equation $\dfrac{dy}{dt} = -5y$, and let $z = y^2$. Then z satisfies a differential equation of the form $\dfrac{dz}{dt} = f(z)$. Find $f(z)$.

For questions 9–14, please show all your work.

9. [8] a) By using a sketch, or otherwise, explain why the equation $e^{-x} = x$ has exactly one solution.

b) In using Newton's Method to find the solution of the equation $e^{-x} = x$, the current estimate is x_n. Find an expression (in terms of x_n) for the next estimate x_{n+1}.

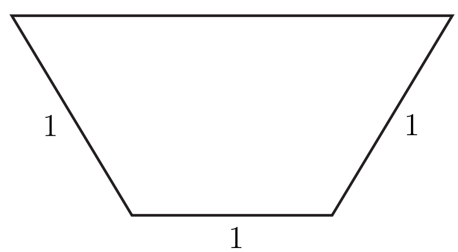
10. [8] Let $f(x) = x^{1/x}$ for $x > 0$.

a) Find $f'(x)$.

b) At what value of x does the curve $y = f(x)$ have a horizontal tangent line?

c) Using the results of parts a) and b) (not a calculator), determine which is larger, $3^{1/3}$ or $\pi^{1/\pi}$. Explain.

11. [10] Three of the sides of a trapezoid have length 1. What should be the length of the fourth side if the area of the trapezoid is to be as large as possible?



12. [10] A function $f(x)$ defined on the whole real line satisfies the following conditions

$$f(0) = 0, \qquad f(2) = 2, \qquad \lim_{x \to +\infty} f(x) = 0,$$
$$f'(x) = k(2x - x^2)e^{-x} \quad \text{for some positive constant } k.$$

a) Determine the intervals on which f is increasing and decreasing and the location of any local maximum and minimum values of f.

b) Determine the intervals on which f is concave upward or downward, and the x-coordinates of any inflection points of f.

c) Determine $\lim_{x \to -\infty} f(x)$.

d) Sketch the graph of $y = f(x)$, showing any asymptotes and the information determined in parts a)–c).

13. [10] The air pressure in an automobile's spare tire was initially 3000 millibar. Unfortunately, the tire had a slow leak. After 10 days the pressure in the tire had declined to 2800 millibar. If $P(t)$ is the air pressure in the tire at time t, then $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = -k(P(t) - A),$$

where k is a constant and A is the atmospheric pressure. For simplicity, take atmospheric pressure to be 1000 millibar. When will the pressure in the tire be 2500 millibar?

14. [10] A water tank has the shape of a vertex-down right circular cone. The depth of the tank is 9 meters, and the top of the tank has radius 6 meters. Water flows into the tank from a hose at a constant rate of 14π cubic metres per hour, and leaks out of a hole at the bottom of the tank at a rate of kh cubic metres per hour when the depth of water in the tank is h metres. Here k is a constant. When the water is 3 metres deep in the tank, its surface is rising at the instantaneous rate of 2 metres per hour. Find the value of the constant k.