SCARRING OF QUASIMODES ON HYPERBOLIC MANIFOLDS

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Abstract. Let $N$ be a compact hyperbolic manifold, $M \subset N$ an embedded totally geodesic submanifold, and let $-\hbar^2 \Delta_N$ be the semiclassical Laplace–Beltrami operator.

For any $\varepsilon > 0$, we explicitly construct families of quasimodes of energy width at most $\varepsilon \frac{h}{|\log h|}$ which exhibit a “strong scar” on $M$; that is, their microlocal lifts converge weakly to a probability measure which places positive weight on $S^*M \hookrightarrow S^*N$. An immediate corollary is that any invariant measure on $S^*N$ occurs in the ergodic decomposition of the semiclassical limit of certain quasimodes of width $\varepsilon \frac{h}{|\log h|}$.

1. Introduction

We consider a problem in the spectral asymptotics of the Laplace–Beltrami operator $\Delta_N$ on a compact Riemannian manifold $N$. Following the “semiclassical” convention we will index our eigenvalues and (approximate) eigenfunctions by a spectral parameter $\hbar$ tending to zero (the corresponding eigenvalue being $\lambda_\hbar = \hbar^{-2}$). Abusing notation we may have $\hbar$ tend to zero along a discrete sequence of values without making this explicit.

As discussed in greater detail below, many results on the concentration behaviour of exact eigenfunctions apply to certain approximate eigenfunctions as well and we address here the converse problem of constructing approximate eigenfunctions with prescribed concentration behaviour. We start by specifying the relevant notion of “approximate eigenfunction”, a relaxation of the eigenfunction equation $\Delta_N \Psi_\hbar = E_\hbar \Psi_\hbar$:

**Definition 1.1.** Fix $C > 0$ and a sequence $\{E_\hbar\}_\hbar$ tending to $E_0 > 0$ (“energies”). A family of quasimodes of width $\frac{Ch}{|\log h|}$ with central energies $E_\hbar$ is a sequence $\{\Psi_\hbar\}_\hbar$ of $L^2(N)$-normalized functions on $N$ such that

$$\|(-\hbar^2 \Delta_N - E_\hbar) \Psi_\hbar\| \leq \frac{Ch}{|\log h|}.$$

If we prefer not to specify $C$ we will use the term “a sequence of log-scale quasimodes”. Essentially without loss of generality we will also fix $E_\hbar = 1$ for all $\hbar$. Finally, note the existence of such a non-zero quasimode as above shows that $-\hbar^2 \Delta_N$ has an eigenvalue $E$ in the interval $\left[ E_\hbar - \frac{Ch}{|\log h|}, E_\hbar + \frac{Ch}{|\log h|} \right]$.

We will study measure-theoretic concentration (more precisely, non-concentration) in the weak-* limit. Precisely, to a quasimode $\Psi_\hbar$ associate the probability measure $\bar{\mu}_\hbar$ on $N$ given by $\bar{\mu}_\hbar(f) = \int_N |\Psi_\hbar|^2 f dV$ for test functions $f$ on $N$, where $dV$ is the Riemannian volume form on $N$.

These measures have natural lifts to distributions $\mu_\hbar$ on the cotangent bundle $T^*N$ (commonly referred to as “microlocal lifts”); we review the construction later. A weak-* limit of
these microlocal lifts is necessarily a probability measure on $T^*N$. These limits, to be denoted $\mu_{sc}$ and called quantum limits or semiclassical measures, are the subject of this paper. Informally, they may be called weak-* limits of the quasimodes themselves.

The reader is advised that these terms (quantum limit and semiclassical measure) are usually reserved for the case where the $\Psi_\hbar$ are exact eigenfunctions, but the more general use is more appropriate here. We will be clear in each invocation of these terms whether we exact eigenfunctions or quasimodes are under consideration.

Perhaps our most striking result is the following

**Theorem (Cor. 1.10).** Let $N$ be a compact hyperbolic manifold (a compact Riemannian manifold of constant negative curvature) then there exists $C > 0$ such that any probability measure on the unit contangent bundle $SN$, invariant under the geodesic flow, arises as the quantum limit of a sequence of log-scale quasimodes of width parameter $C$.

In the next part of the introduction we motivate our work by reviewing the quantum unique ergodicity problem and results toward it. Knowledgeable readers may wish to skip to Section 1.3 where we discuss all our results.

### 1.1. The quantum unique ergodicity problem.

The cotangent bundle $T^*N$ is naturally the phase space for a single particle moving on our manifold $N$. We fix a quantization scheme $\text{Op}_\hbar$, assigning to each observable $a$ a smooth function on $T^*N$ belonging to an appropriate symbol class an operator $\text{Op}_\hbar(a): L^2(N) \to L^2(N)$. Fix a positive observable $H$ (called it the Hamiltonian), and suppose that for a sequence of values of $\hbar$ tending to zero we have chosen a corresponding sequence of normalized eigenfunctions $\Psi_\hbar \in L^2(N)$ where

$$\text{Op}_\hbar(H)\Psi_\hbar = E_\hbar\Psi_\hbar. \quad (1.2)$$

We suppose $E_\hbar = E_0 + \mathcal{O}(\hbar)$ for some fixed $E_0 > 0$. To each $\Psi_\hbar$ we associate its Wigner measure, the distribution $\mu_\hbar$ on $T^*N$ given by

$$\mu_\hbar(a) = \langle \Psi_\hbar, \text{Op}_\hbar(a)\Psi_\hbar \rangle.$$

A major problem in spectral asymptotics (or mathematical physics) is to study the concentration of eigenfunctions $\Psi_\hbar$ – the basic expectation is that the more chaotic the classical dynamics induced by $H$, the more uniformly distributed the eigenfunctions $\Psi_\hbar$ are as $\hbar \to 0$. One tool for studying this problem is examining subsequential weak-* limits of the $\mu_\hbar$, the “semiclassical measures” or “quantum limits” $\mu_{sc}$. Their study was initiated in the works of Schnirel’man, Zelditch, and Colin de Verdi`ere [26, 28, 9] with the following observations:

First, since two different quantization schemes differ by terms of order $\mathcal{O}(\hbar)$, the measure $\mu_{sc}$ depends on the sequence $\Psi_\hbar$ but not on the scheme itself. Second, the existence of a positive quantization scheme (the so-called Friedrich symmetrization) shows that any limit must be a positive measure. Using for $a$ the constant function 1 – for which $\text{Op}_\hbar(a)$ is the identity operator – shows that our limit $\mu_{sc}$ is a probability measure. Standard techniques also show that the measure $\mu_{sc}$ must be supported on the energy surface $H(x, \xi) = E_0$.

Third, Egorov’s Theorem relates the Hamiltonian flow induced by $H$ to the action of the Schrödinger propagator $U(t) = \exp\left(-\frac{i}{\hbar} \text{Op}_\hbar(H)\right)$. Since replacing $\Psi_\hbar$ with its time-evolved state $U(t)\Psi_\hbar = \exp\left(-\frac{itE_0}{\hbar}\right)\Psi_\hbar$ has no effect on $\mu_\hbar$ one can show that any limit $\mu_{sc}$ must
be invariant under the Hamiltonian flow generated by $H$. For a more general discussion of these properties, see the book [30].

**Problem 1.3 (Quantum Unique Ergodicity (“QUE”)).** Classify, amongst the flow-invariant measures supported on level sets of $H$, those which are weak-* limits of sequences of Wigner measures of eigenfunctions.

The primary case is of a free particle moving on a compact Riemannian manifold, that is where the classical dynamics is geodesic flow. The Hamiltonian is $H(x, ξ) = \|ξ\|_g^2$ (we identify the tangent and cotangent bundles using the metric $g$), whose quantization $\text{Op}_\hbar(H)$ is $-\hbar^2 Δ_g$, where $Δ_g$ is the Laplace–Beltrami operator for the metric $g$. The eigenfunction equation (1.2) is then the familiar spectral problem $ΔΨ = -λΨ$ where we take $\hbar = \sqrt{\frac{1}{λ}}$ and $E = 1$.

For non-zero $E$, the energy surfaces $H = E$ are then all equivalent to the unit tangent bundle $SM$ (our energy surface). The induced metric there gives rise to a natural geodesic-flow invariant measure, Liouville measure $μ_L$.

The genesis of Problem 1.3 is the main Theorem of [26, 28, 9]. We revert here to indexing the eigenfunctions of $Δ_N$ in numerical order with an integer parameter $n$ since the result concerns the entire spectrum:

**Theorem 1.4.** Let $N$ be a compact Riemannian manifold, let $\{Ψ_n\}_{n=0}^\infty ⊂ L^2(N)$ be a complete eigenbasis for the positive Laplace–Beltrami operator with corresponding eigenvalues $\{λ_n\}_{n=0}^\infty$. Write $μ_n$ for the Wigner measure associated to the eigenfunction $Ψ_n$.

1. (Convergence on average) The following statement on distributions holds:

$$\frac{1}{T} \sum_{n<T} μ_n \overset{wk-*}{\longrightarrow} μ_L.$$

2. (“Quantum ergodicity”) If the Liouville measure $μ_L$ is ergodic with respect to the geodesic flow, then there exists a subsequence $\{Ψ_{n_k}\}_{k=0}^\infty$ of density one along which the Wigner measures converge to $μ_L$. Stated otherwise, almost all eigenfunctions asymptotically equidistribute on $SN$.

A major case where the Liouville measure is ergodic is that of negatively curved manifolds where we further have

**Conjecture 1.5 (Quantum Unique Ergodicity; Rudnick–Sarnak [23]).** For $N$ compact of negative sectional curvature, we have $μ_n \overset{wk-*}{\longrightarrow} μ_L$. In other words, Liouville measure is the unique quantum limit.

This conjecture predicts uniform distribution (in the root-mean-square sense) of the eigenfunctions. The opposite behaviour (some enhancement of some eigenfunctions along closed geodesics) was observed numerically by Heller [17] in the case of plane billiards and given then name “scarring”. Perhaps the strongest form of this phenomenon, a “strong scar” along an invariant measure, is the situation of a semiclassical measure having this singular measure as an atom in its ergodic decomposition.\footnote{Formally, this holds up to an operator of order $O_{L^2}(h)$ – perhaps it is better to use the converse formulation that the principal symbol of $-\hbar^2 Δ_g$ is $|ξ|^2_g$.}
Some ergodic Euclidean billiards were shown to be Quantum Ergodic in the sense of Theorem 1.4 by Gérard–Leichtnam [15] and Zelditch–Zworski [29]). The question of whether the numerically observed scarring persists in the semiclassical limit was mostly settled by Hassel, who showed in [16] that for the one-parameter family of billiards known as the Bunimovich stadium, for almost every member there is a sequence of eigenfunctions whose quantum limit puts positive mass on the set of “bouncing ball” trajectories, a one-parameter family of closed geodesics. In negative curvature closed trajectories are unstable, so such families cannot exist.

Positive results toward the Rudnick–Sarnak Conjecture were first famously obtained by Lindenstrauss [21] in the case of hyperbolic surfaces with the surface and the eigenfunctions enyoing additional arithmetic symmetries (the arguments were recently simplified by Brooks–Lindenstrauss in [8]). Positive results on general manifolds all depend on the breakthrough of Anantharaman [2]. She showed that for manifolds with geodesic flows having the Anosov property, quantum limits must have positive entropy with respect to the action of the geodesic flow, which in particular rules out the possibility of very singular measures being quantum limits (a measure supposed on a union of closed geodesics, for example). Follow-up works with generalizations and improvements include [5, 4, 6, 3, 22].

The present article supports this line of work – specifically to studying the extent to which these positive-entropy results are sharp.

1.2. Quasimodes. Many of the positive-entropy results discussed in the previous section continue to hold when one weakens the hypothesis that the $\Psi_\hbar$ are exact eigenfunctions. In fact, the hypothesis that they are log-scale quasimodes in the sense of Definition 1.1 suffices. For these we have:

1. Every quantum limit of a sequence of log-scale quasimodes is a probability measure supported on the energy surface $SN$.
2. This measure is invariant under the geodesic flow flow.
3. On a manifold of negative sectional curvature, any weak-* limit of log-scale quasimodes with $C > 0$ small enough has positive entropy.

**Problem 1.6** (QUE for quasimodes). Classify the weak-* limits of Wigner measures associated to log-scale quasimodes.

This version of Anantharaman’s result is quantitative, in that the entropy bound depends on the spectral width parameter $C$. Conversely one may ask what is the largest possible entropy of a quantum limit of a sequence of quasimodes of such width.

The first such result was due to Brooks:

**Theorem 1.7** ([7]). Let $N$ be a compact hyperbolic surface, and let $\gamma \subset N$ be a closed geodesic. Then for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ and a sequence of $\frac{\varepsilon h}{\log h}$-quasimodes and central energies $E_\hbar$ converging to a semiclassical measure $\mu_{sc}$ with $\mu_{sc}(\{S\gamma\}) \geq \delta(\varepsilon)$.

Brooks’s construction is analogous to one of Faure–Nonnenmacher–De Bièvre [14] in the toy model known as the “quantum cat map”. In particular, the result depends on the periodic boundary conditions on the hyperbolic surface, and on the connection between eigenfunctions of the Laplace operator and the representation theory of $\text{SL}_2(\mathbb{R})$.

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2The constant $C$ may vary from one sequence to another.
Using microlocal techniques instead, Nonnenmacher and the first named author obtained a result for general Hamiltonians $H$ on a surface:

**Theorem 1.8 ([13]).** Let $(N, g)$ be a compact Riemannian surface, and let $\gamma$ be a hyperbolic orbit of the Hamiltonian flow on an energy surface $H^{-1}(E_0)$ for some energy $E_0 > 0$. Then for any $\varepsilon > 0$ is a sequence of quasimodes of spectral width at most $\varepsilon \frac{\hbar}{\log \hbar}$ and central energies $E_{\hbar} = E_0 + O(\hbar / \log \hbar)$ of the quantum Hamiltonian $Op_{\hbar}(H)$ whose weak-* limit gives $S_{\gamma}$ at least $\varepsilon \frac{2}{3 \sqrt{3}} + O((\varepsilon / \lambda_{\gamma})^2)$ mass. Here $\lambda_{\gamma}$ is the expansion rate along the unstable direction of the orbit $\gamma$.

**1.3. Results of this paper.** We seek to extend the above results to higher-dimensional manifolds, replacing the periodic geodesic $\gamma$ with a totally geodesic submanifold $M$, and achieve this goal when $N$ is a hyperbolic manifold (constant negative curvature, as in Brooks’s result). However, our techniques are mostly microlocal and there is cause to hope that they would apply in general.

The result is best thought of as a “transfer” principle: given a sequence of log-scale quasimodes on $M$ with associated quantum limit $\mu_M$, we extend them transversely using the hyperbolic dynamics transverse to $M$ to log-scale quasimodes on $N$ which still concentrate on $SM$ in the same manner as the original sequence (or at least give it positive mass). To obtain specifically the uniform measure on $M$, we use the equidistributed sequence provided by the Quantum Ergodicity Theorem. Our main result is:

**Theorem 1.9.** Let $N$ be a complete hyperbolic manifold, let $M \subset N$ be an embedded compact totally geodesic submanifold, and pick a central energy $E_0 > 0$.

1. Suppose given a sequence of central energies energies $E_{\hbar} = E_0 + O(\hbar)$ and some $\delta > 0$. Select a width constant which satisfies $\pi \tilde{\lambda}_{E_0}(1 + \delta) \geq C = C_M \geq \pi \tilde{\lambda}_{E_0}$ where $\tilde{\lambda}_{E_0} = 2\sqrt{E_0}$ is the expansion rate in the unstable directions transverse to $M$. Then there exists a sequence of log-scale quasimodes of $N$ with the given spectral data defined for $\hbar \leq \hbar_0(M, \delta)$ and whose weak-* limit is the Liouville measure on $SM$, namely $\delta_{SM}$.

2. Suppose in addition that $N$ is compact. Then for any $\varepsilon > 0$ there exists a sequence of $\frac{\varepsilon \hbar}{\log \hbar}$-quasimodes with central energies converging to $E_0$ as above, whose weak-* limit has $\delta_{SM}$ as an ergodic component of mass at least $\eta = \eta(C_M, \varepsilon) > 0$.

3. In both cases one may replace $\delta_{SM}$ with any log-scale semiclassical measure on $M$ at the cost of widening the quasimodes to the sum of the spectral width reported in (1),(2) respectively and the width of the quasimodes on $M$.

The bulk of the paper is devoted to establishing (1). [13, Sec. 6] shows how to deduce (2) from (1) using spectral projection. As detailed in Lemma 4.5, the same arguments automatically establish (3) as well.

Unlike the previous results for surfaces, our proof avoids relying on the dimensions of $N$ and $M$ or on the relative dimension of $N$ in $M$.

We expect the natural higher-dimensional generalization of Theorem 1.8 holds, with the only assumption that the dynamics transverse to $M$ is hyperbolic. That motivates our statement of the technical result, since at that level of generality no better statement seems possible. It turns out, however, that under our assumption of negative curvature the case of closed geodesics (exactly the totally geodesic 1-dimensional submanifolds) suffices – and
in fact can be used to obtain more: we can go beyond log-scale semiclassical measures on totally geodesic submanifolds and obtain every invariant measure whatever:

**Corollary 1.10.** Let $N$ be a compact hyperbolic manifold, and let $\mu$ be a probability measure on $SN$ invariant by the geodesic flow. Then there is a sequence of log-scale quasimodes whose associated semiclassical measure is $\mu$. Further, given $\varepsilon > 0$ there is $\eta = \eta(\varepsilon) > 0$ and a sequence of $\frac{ch}{|\log h|}$-width quasimodes on $N$ whose weak-$^*$ limit $\mu_\infty$ carries weight at least $\eta(\varepsilon)$ on the component $\mu$.

**Proof.** It was shown by Sigmund [24] that there is a sequence $\{\gamma_k\}_{k=1}^\infty$ of periodic geodesics whose natural measures $\delta_k = \delta_{S\gamma_k}$ converge in the weak-$^*$ topology to $\mu$. For each $k$ let $\{\psi_{k,n}\}_{n=1}^\infty$ be a sequence of log-scale quasimodes on $N$ (all sequences of the same spectral width) whose weak-$^*$ limit is $\delta_{S\gamma_k}$ guaranteed by Theorem 1.9 (1). By the separability of the space $C(SN)$ dual to the space of measures, a diagonal argument – using the uniformity in the constants from Theorem 1.9 (1) – gives a subsequence $\{\psi_{k_i,n_i}\}_{i=1}^\infty$ which is a sequence of log-scale quasimodes and which converges to the desired limit $\mu$. The spectral projection argument can again be used to shorten the spectral width to $\frac{ch}{|\log h|}$ while keeping at least weight $\eta(\varepsilon)$ on the component $\mu$. \qed

To the knowledge of the authors, this is the first result in the mathematical quantum chaos literature which demonstrates that $o(h)$ quasimodes (which yield invariant semiclassical measures, log-scale or not) can develop scars on fractal subsets. As in the work of Brooks our bounds explore the extend to which entropy bounds on quasimodes are sharp, though in both cases the lower and upper bounds are far from meeting.

It is important to add that the construction of quasimodes which localize along closed geodesics, or more generally along smooth invariant submanifolds, has a long and rich history. For a brief exposition of this history, see the introduction of [13] and the references therein.

1.4. **Outline of the proof and further remarks.** The main idea of our proof is to combine the long-time evolution idea first originated in the work of Vergini-Schneider [27] and the Fermi-normal coordinate/quantum Birkhoff normal idea of Colin de Verdière-Parisse [10].

Let $N$ be a hyperbolic manifold, $M \subset N$ a compact totally geodesic submanifold. Then for $\varepsilon_1 > 0$ small enough, the universal cover of the $\varepsilon_1$-neighbourhood $N_{\varepsilon_1}(M) \subset N$ (thought of as a subset of the universal cover $\tilde{N}$) is exactly the $\varepsilon_1$-neighbourhood $N_{\varepsilon_1}(\tilde{M})$. In particular, we have a well-defined nearest-neighbour projection $\pi: N_{\varepsilon_1}(M) \to M$ and a co-ordinate system $z \mapsto (\pi(z), u)$ identifying $N_{\varepsilon_1}(M)$ with a product $M \times B$ where $B$ is a Euclidean $\varepsilon_1$-ball dimension $r = \dim N - \dim M$. In this coordinate the metric takes the form of a warped product (here we use our hypothesis that $N$ is hyperbolic, by using the precise form of the metric), so that separating the variables gives the formula

$$\Delta_N = \frac{1}{1 + |u|^2} \Delta_M + \Delta_u + |u|^2 \frac{\partial^2}{\partial u^2} + nu \cdot \frac{\partial}{\partial u}. \quad (1.11)$$

Here $\Delta_M$ is the Laplace–Beltrami operator on $M$ a hyperbolic manifold, $\Delta_u$ is the Euclidean Laplace–Beltrami operator $\sum_i \frac{\partial^2}{\partial u_i^2}$ and $u \frac{\partial}{\partial u}$ is a radial differentiation operator. The calculation is reproduced in Section 2.2.
For our quasimode we use the ansatz

$$\Psi_h(z) = \psi_h(u) \varphi_h(\pi(z))$$

where $\varphi_h$ is an eigenstate on $M$ of energy $E$ (so that $\varphi_h$ is stationary for the Schrödinger propagator associated to $\Delta_M$). $\Psi_h$ will concentrate on $M$ exactly when $\psi_h$ will concentrate on the origin of the $u$-plane.

This improves on the construction in [13] where the $\psi_h$ were constructed from localized wave packets on $M$, so that their evolution (and especially self-interference) was non-trivial and needed to be considered as well. Our use of exact eigenfunctions (more generally, of quasimodes of specified width) simplifies the construction.

Plugging into the Schrödinger equation

$$-\hbar^2 \Delta_N \Psi_h \approx E \Psi_h$$

we see that $\psi_h$ needs to be a quasimode of energy 0 for the Schrödinger operator

$$-\hbar^2 \Delta_u - \hbar^2 u^2 \frac{\partial^2}{\partial u^2} - nth^2 u \frac{\partial}{\partial u} - \frac{Eu^2}{1+u^2}.$$

concentrating near the origin. The term $nth^2 u \frac{\partial}{\partial u}$ is lower order in $\hbar$ and hence negligible. Ignoring for the moment the mixed term $\hbar^2 u^2 \frac{\partial^2}{\partial u^2}$ and approximating $\frac{u^2}{1+u^2} \approx u^2$ (our wave packets will be concentrated near $u = 0$) – we have approximately the operator

$$-\hbar^2 \Delta_u - Eu^2,$$

known as the inverted harmonic oscillator – and it is well-known that this Hamiltonian has log-scale quasimodes concentrating at the origin (which is a hyperbolic fixed point).

To deal with our somewhat different Hamiltonian we employ a Quantum Birkhoff Normal Form due to Iantchenko [19]. This transformation (given by a Fourier integral operator) approximates our Schrödinger operator near its hyperbolic fixed point by an inverted Harmonic oscillator together with controlled higher-order corrections.

This approach is not a novelty, but both $u$ varying in $\mathbb{R}^r$ rather than $\mathbb{R}$ and our desire for uniformity sufficient for the argument of Corollary 1.10 require working through the arguments with some care.

Remark 1.12. While we sometimes use the fact that all Lyapunov exponents are equal, the argument in the transverse direction (using the quantum Birkhoff normal form) should hold generally.

There is then reason to hope that our result could be extended to the case where $N$ is a general manifold and $M \subset N$ is an invariant hyperbolic submanifold (that is, a totally geodesic submanifold where the dynamics transverse to $M$ are uniformly expanding and contracting). The difficulty in establishing that case is that the exact separation of variables available in constant negative curvature would only hold approximately. The transverse operator would be more complicated (and less explicit) and further work would be required to establish the existence of a Birkhoff normal form.

Remark 1.13. While we managed to construct log-scale quasimodes concentrating on general invariant measures (including fractal ones) it would be interesting to construct them directly without appealing to Sigmund’s Theorem. This would allow us to better explore the quantitative relation between the spectral width of the quasimodes and the possible
entropies of the associated quantum limits: even for measures of relatively large entropy we require sufficient spectral width to obtain concentration on closed geodesics.

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2. Coordinates and Decompositions

In this section we discuss the hyperbolic geometry from the symmetric space point of view and provide an explicit Fermi normal coordinate system near our selected totally geodesic submanifold \(M\), which will allow us to re-write the operator \(\hbar^2 \Delta_N\) with respect to a warped-product structure. This is the higher dimensional analogue of the idea in Section 6 of the work of Colin de Verdière-Parisse [10].

**Remark 2.1.** Identifying the tangent and cotangent bundles using the Riemannian metric, we may use the notations \(TN\) and \(T^*N\) (along with those for natural sub-bundles) interchangeably.

2.1. A non-quantitative collar Lemma. Our setup is as follows:

- Let \(G\) be a semisimple Lie group with Iwasawa decomposition \(G = NAK\), \(H < G\) be a semisimple closed subgroup containing \(A\) such that \(K_H = K \cap H\) is a maximal compact subgroup of \(H\).
- Let \(S_H = H/K_H\) and \(S = G/K\) be the corresponding symmetric spaces; note that \(S_H\) embeds in \(S\) as a totally geodesic submanifold.
- Let \(\Gamma < G\) be a discrete subgroup such that \(\Gamma_H = \Gamma \cap H\) is a uniform lattice in \(H\).

**Lemma 2.2.** After passing to a finite-index subgroup we may assume \(\Gamma_H \setminus (\Gamma \cap HK) = \{1\}\).

**Proof.** Let \(\mathcal{F}_H\) be a fundamental domain for \(\Gamma_H \setminus H\). Then the finite set \(\Gamma \cap \mathcal{F}_HK\) is a set of representatives for the quotient. Since \(\Gamma\) is residually finite there is a subgroup \(\Gamma^1\) not containing these elements (except for the identity), so that such that \(\Gamma^1 \cap HK = \Gamma^1 \cap H = \Gamma^1_H\). \(\square\)

**Assumption 2.3.** For the rest of this article, we let \(M = \Gamma_H \setminus H/K_H\) be embedded in \(N = \Gamma \setminus G/K\).

Now, let \(\pi: S \to S_H\) be the projection on the convex subset, and write \(N_\varepsilon(S_H)\) to be the \(\varepsilon\)-neighbourhood of \(S_H\) in \(S\).

**Lemma 2.4.** There exists an \(\varepsilon\) such that if \(\gamma \in \Gamma\) has \(\gamma N_\varepsilon(S_H) \cap N_\varepsilon(S_H)\) then \(\gamma \in \Gamma_H\).

**Proof.** If not there are \(\gamma_n \in \Gamma\) not in \(\Gamma_H\) and \(z_n \in S\) such that \(z_n, \gamma_n z_n \in N_{\varepsilon_n}(S_H)\) with \(\varepsilon_n \to 0\). Let \(y_n = \pi(z_n), y'_n = \pi(\gamma_n z_n)\). Let \(\lambda_n, \lambda'_n \in \Gamma_H\) be such that \(\lambda_n y_n, \lambda'_n y'_n \in \mathcal{F}_H/K_H\). Then replacing \(z_n\) with \(\lambda_n z_n\) and \(\gamma_n\) with \(\lambda_n \gamma_n \lambda_n^{-1}\) we may assume that \(y_n, y'_n \in \mathcal{F}_H/K_H\) (and still \(\gamma_n \notin \Gamma_H\)). Passing to a subsequence we may assume that \(y_n \to y_\infty, y'_n \to y'_\infty\). We also have \(d(z_n, y_n) = d(\gamma_n z_n, y'_n) \leq \varepsilon_n \to 0\) so also \(z_n \to y_n, \gamma_n z_n \to y_n\). In particular
\( \gamma_n \) move \( y_{\infty} \) a bounded amount so belong to a finite set, and we may further assume the sequence is a constant element \( \gamma \) such that \( \gamma y_n = y'_n \). But then \( \gamma \in \Gamma \cap HK \) so \( \gamma \in \Gamma_H \) as claimed.

**Corollary 2.5** (Collar Neighborhood). For \( \varepsilon \) small enough, \( N_\varepsilon (M) = \Gamma_H \setminus N_\varepsilon (S_H) \).

### 2.2. Hyperbolic space.

Suppose now \( G = O(n, 1) \supset O(m, 1) = H \) and write \( r = n - m \). For \( S \) take the upper half-space model with coordinates \((y; x_1, \ldots, x_{m-1}, w_m, \ldots, w_{n-1})\) so that \( S_H = \{(y; x, \Omega)\} \).

Given a point \( z = (y; x, w) \in S \) let \( \pi(z) \in S_H \) be the nearest-neighbour projection. Writing \( w \) in spherical coordinates \((w, \Omega)\) with \( w = |w|, \Omega = \frac{w}{|w|} \), the projection \( \pi(z) \) is the point \( (\eta; x) \) where \( \eta = \sqrt{y^2 + w^2} \). Writing \( u = \frac{w}{y} \), the hyperbolic distance between \( z \) and \( \pi(z) \) is

\[ \rho = \rho(z, \pi(z)) = \arcsinh u = \log \left( u + \sqrt{1 + u^2} \right). \]

We note that \(|u| = \sinh \rho, \sqrt{1 + |u|^2} = \tanh \rho \).

Hence \((\eta, x, \rho, \Omega)\) is a coordinate system for \( S \) with the inverse map being \( y = \frac{\eta}{\cosh \rho}, \quad w = \eta \tanh \rho, \quad \Omega = \eta \tanh \rho \Omega \). We compute the Laplace operator in these coordinates. For this recall that the Laplace-Beltrami operator on hyperbolic \( n \)-space in the upper half-space model is

\[ \left( \frac{\partial}{\partial y} \right)^2 - (n-1) \left( \frac{\partial}{\partial y} \right) + y^2 (\Delta_x + \Delta_w) \]

where \( \Delta_x = \sum_i \frac{\partial^2}{\partial x_i^2} \) and similarly for \( w \). We note for reference that \( \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1+|u|^2}} \) so that \( \frac{\partial \rho}{\partial y} = \frac{1}{\sqrt{1+|u|^2}} \frac{\partial u}{\partial y} \) and similarly for \( w \) instead of \( y \).

By the Collar Lemma we may use the Fermi normal coordinate system established above for \( S \) locally in \( N_\varepsilon (M) \), getting:

**Lemma 2.6.** In the collar neighborhood described in (2.5), the Laplace–Beltrami operator has the following Fermi-normal coordinate-type structure:

\[ \Delta_N = \frac{1}{1 + |u|^2} \Delta_M + \Delta_u + |u|^2 \left( \sum_{i=1}^{r} \frac{u_i}{|u|^2} \frac{\partial}{\partial u_i} \right)^2 + n \left( \sum_{i=1}^{r} u_i \frac{\partial}{\partial u_i} \right) \]

where \( u \in \mathbb{R}^r \). Furthermore, the volume element on \( N \) in these coordinates is

\[ \left( \eta^{-m} \, d\eta \, d^{m-1} \eta \right) \left( (1 + |u|^2) \frac{m-1}{2} \, du \right), \]

where the first term is the volume element of the hyperbolic metric on \( M \).

**Proof.** We give only the main parts of the calculation and leave the intermediate steps to the reader.

We have

\[ \frac{\partial}{\partial y} = \frac{1}{\cosh \rho} \frac{\partial}{\partial \eta} - \frac{1}{y} \tanh \rho \frac{\partial}{\partial \rho}. \]

Using \( y = \frac{\eta}{\cosh \rho} \) gives

\[ y \frac{\partial}{\partial y} = \frac{1}{\cosh^2 \rho} H - (\tanh \rho) R \]
where we have set $H = \eta \frac{\partial}{\partial \eta}$ and $R = \frac{\partial}{\partial \rho}$. Similarly we have
\[
\frac{\partial}{\partial w} = \frac{1}{\eta} ((\tanh \rho) H + R).
\]
Now,
\[
\left( y \frac{\partial}{\partial y} \right)^2 = \frac{1}{\cosh^4 \rho} H^2 + \frac{2 \sinh^2 \rho}{\cosh^2 \rho} H - \frac{2 \sinh \rho}{\cosh^2 \rho} H R + \frac{\sinh^2 \rho}{\cosh^3 \rho} R^2 + \frac{\sinh \rho}{\cosh^3 \rho} R
\]
and
\[
\left( \frac{\partial}{\partial w} \right)^2 = \frac{1}{\eta^2} \left( \frac{\sinh^2 \rho}{\cosh^2 \rho} H^2 + \frac{1 - \sinh^2 \rho}{\cosh^2 \rho} H + \frac{2 \sinh \rho}{\cosh \rho} H R + R^2 - \frac{\sinh \rho}{\cosh \rho} R \right).
\]
Finally, the Euclidean Laplace operator in polar coordinates reads
\[
\Delta_w = \frac{\partial^2}{(\partial w)^2} + \frac{r - 1}{w} \frac{\partial}{\partial w} + \frac{1}{w^2} \Delta_{S^{r-1}}.
\]
A tedious calculation then gives
\[
\Delta_N = \frac{1}{\cosh \rho} \Delta_M + K + \frac{1}{\sinh \rho} \Delta_{S^{r-1}},
\]
with $K = R^2 + \left( (n - 1) \tanh \rho + \frac{r - 1}{\sinh \rho \cosh \rho} \right) R$.

Making the change of variables $|u| = \sinh \rho$ for which $\frac{\partial}{\partial \rho} = \sqrt{1 + u^2} \frac{\partial}{\partial u}$ leads to
\[
K + \frac{1}{\sinh^2 \rho} E = \Delta_u + |u|^2 \frac{\partial^2}{\partial u^2} + n \left( \sum_{i=1}^{r} u_i \frac{\partial}{\partial u_i} \right).
\]

3. Semiclassical preliminaries

In this section we recall the concepts and definitions from semiclassical analysis required for the sequel. Notations are drawn from the monographs [30].

3.1. Pseudodifferential operators on a manifold. Recall the standard classes of symbols on $\mathbb{R}^{2d}$
\[
S^m(\mathbb{R}^{2d}) \overset{\text{def}}{=} \left\{ a \in C^\infty(\mathbb{R}^{2d} \times (0, 1]) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} |\xi|^{m-|\beta|} \right\}.
\] (3.1)
Symbols in this class can be quantized through the $\hbar$-Weyl quantization into the following pseudodifferential operators acting on $u \in S(\mathbb{R}^d)$:
\[
\text{Op}_h^w(a) u(x) \overset{\text{def}}{=} \frac{1}{(2\pi \hbar)^d} \int_{\mathbb{R}^{2d}} e^{i \frac{1}{\hbar} \langle x-y, \xi \rangle} a \left( \frac{x+y}{2}, \xi; \hbar \right) u(y) dy d\xi.
\] (3.2)
One can adapt this quantization procedure to the case of the phase space $T^*M$, where $M$ is a smooth compact manifold of dimension $d$ (without boundary) by performing the Fourier analysis in local coordinates.

In more detail, cover $M$ with a finite set $(f_l, V_l)_{l=1,\ldots,L}$ of coordinate charts, where each $f_l$ is a smooth diffeomorphism from $V_l \subset M$ to a bounded open set $W_l \subset \mathbb{R}^d$. To each $f_l$
correspond a pullback $f_t^* : C^\infty(W_i) \to C^\infty(V_i)$ and a symplectic diffeomorphism $\tilde{f}_t$ from $T^*V_i$ to $T^*W_i$:

$$\tilde{f}_t : (x, \xi) \mapsto \left( f_t(x), (Df_t(x)^{-1})^T \xi \right).$$

Choose a smooth partition of unity $(\varphi_l)_l$ adapted to our cover. That means $\sum_l \varphi_l = 1$ and $\varphi_l \in C^\infty(V_i)$. Then, any observable $a$ in $C^\infty(T^*M)$ can be decomposed as $a = \sum_l a_l$, where $a_l = \alpha \varphi_l$. Each $a_l$ belongs to $C^\infty(T^*V_i)$ and can be identified with the function $\tilde{a}_l = (\tilde{f}_l^{-1})^*a_l \in C^\infty(T^*W_i)$. We now define the class of symbols of order $m$ on $T^*M$ (slightly abusing notation by treating $(x, \xi)$ as coordinates on $T^*W_i$) via

$$S^m(T^*M) \overset{\text{def}}{=} \{ a \in C^\infty(T^*M \times (0, 1]) : a = \sum_l a_l, \text{ such that } \tilde{a}_l \in S^m(\mathbb{R}^d) \text{ for each } l \}. $$

This class is independent of the choice of cover and partition of unity. For any $a \in S^m(T^*M)$, one can associate to each component $\tilde{a}_l \in S^m(\mathbb{R}^d)$ its Weyl quantization $\text{Op}_h^w(\tilde{a}_l)$, which acts on functions on $\mathbb{R}^d$. To get back to operators acting on $M$, we choose smooth cutoffs $\psi_l \in C_c^\infty(V_i)$ such that $\psi_l = 1$ close to the support of $\varphi_l$, and set

$$\text{Op}_h(a)u = \sum_l \psi_l \times (f_t^* \text{Op}_h^w(\tilde{a}_l)(f_t^{-1})^*)(\psi_l \times u), \ u \in C^\infty(M).$$

This quantization procedure maps (modulo smoothing operators with seminorms $O(\hbar^\infty)$) symbols $a \in S^m(T^*M)$ onto the space $\Psi^m_h(M)$ of semiclassical pseudodifferential operators of order $m$. The dependence in the cutoffs $\varphi_l$ and $\psi_l$ only appears at order $\hbar \Psi^m_{\hbar^{-1}}$ (Theorem 18.1.17 [18] or Theorem 9.10 [30]), so that the principal symbol map $\sigma_0 : \Psi^m_h(M) \to S^m(T^*M)/\hbar S^{m-1}(T^*M)$ is intrinsically defined. Most of microlocal calculus on $\mathbb{R}^d$ (for example the composition of operators, the Egorov and Calderón-Vaillancourt Theorems) then extends to the manifold case.

An important example of a pseudodifferential operator is the semiclassical Laplace–Beltrami operator $P(\hbar) = -\frac{1}{2}\Delta_g$. In local coordinates $(x; \xi)$ on $T^*M$, the operator can be expressed as $\text{Op}_h^w \left( |\xi|^2 + \hbar \sum_j b_j(x)\xi_j + c(x) + \hbar^2 d(x) \right)$ for some functions $b_j, c, d$ on $M$. In particular, its semiclassical principal symbol is the function $|\xi|^2_{\hbar} \in S^2(T^*M)$. Similarly, the principal symbol of the Schrödinger operator $-\frac{1}{2}\Delta_g + V(x)$ (with $V \in C^\infty(M)$) is $|\xi|^2_{\hbar} + V(x) \in S^2(T^*M)$.

We will need the slightly more general class of symbols used in [12] (in fact, its adaption to $T^*M$ via the partition of unity argument above). In Euclidean space this is the class given for any $0 \leq \delta < 2/3$ by

$$S^m_{\delta}(\mathbb{R}^d) \overset{\text{def}}{=} \{ a \in C^\infty(\mathbb{R}^d \times (0, 1]) : |\partial_x^\alpha \partial_{\xi}^\beta a| \leq C_{\alpha, \beta} \hbar^{-|\alpha| + |\beta|} |\xi|^{m-|\beta|} \}. $$

These symbols are allowed to oscillate more strongly when $\hbar \to 0$. All the previous remarks regarding the case of $\delta = 0$ transfer over in a straightforward manner.

4. Ansatz and Quantum Birkhoff Normal Forms

4.1. Separation of variables and the derivation of a hyperbolic operator. At this point, we must fix our Fermi collar neighborhood (2.5) and therefore fix its corresponding width $\varepsilon_1 > 0$ (the subscript is intentional, for reasons to be explained later). In spite of our construction being entirely local, we will later show that it extends globally therefore making
our norm estimates and spectral-width estimates independent of the chosen neighborhood $N_{\varepsilon_1}(M)$, after choosing our spectral parameters $\hbar \leq \hbar_0(M, \varepsilon_1)$.

For the problem of finding sufficiently thin spectral-width quasimodes whose semiclassical measures place mass on the energy surface $S_{E_0} M$, we consider the following natural ansatz in our coordinate system of Lemma 2.6:

$$\Psi_h(x, \eta, u) = \psi_h(u) \varphi_h(x, \eta). \quad (4.1)$$

For the moment choose $\varphi_h$ to be an exact eigenfunction of $\Delta_M$, in that $-\hbar^2 \Delta_M \psi_h = E_0 \psi_h$ for some $E_0 > 0$, and the problem is to choose $\psi_h(u)$. Our quasi-eigenvalues (=central energies) will have the form $E_h = E_0 + f(h)$ where $f(h) = O(h)$ so we need to consider the quantity

$$(-\hbar^2 \Delta_N - E_h) \Psi_h(x, \eta, u) \quad (4.2)$$

whilst keeping in mind the operator from the same lemma. The above expression then takes the form

$$-\hbar^2 \left( \Delta_u + |u|^2 \left( \sum_{i=1}^r \frac{u_i}{|u|^2} \frac{\partial}{\partial u_i} \right)^2 + n \left( \sum_{i=1}^r \frac{u_i}{|u|^2} \frac{\partial}{\partial u_i} \right) - \frac{E_0 u^2}{1 + u^2} + f(h) \right) \varphi(u) \psi(x, \eta). \quad (4.3)$$

We therefore isolate the $u$-variable operator

$$K_u(h) \overset{\text{def}}{=} -\hbar^2 \left( \Delta_u + |u|^2 \left( \sum_{i=1}^r \frac{u_i}{|u|^2} \frac{\partial}{\partial u_i} \right)^2 + n \left( \sum_{i=1}^r \frac{u_i}{|u|^2} \frac{\partial}{\partial u_i} \right) - \frac{E_0 u^2}{1 + u^2} + f(h) \right). \quad (4.4)$$

The factorization of the volume element in Lemma 2.6 shows we should consider $K_u(h)$ acting on $L^2(\mathbb{R}^r, (1 + u^2)^{-\frac{m-1}{2}} du)$, where it will be a formally symmetric operator (on smooth compactly supported functions) by virtue of $\Delta_N$ being symmetric.

**Lemma 4.5.** Suppose that $\{\varphi_h\}_h$ are normalized quasimodes for $-\hbar^2 \Delta_M$ with central energies $E_h = E_0 + f(h)$ and of width $\frac{c_1 h}{|\log h|}$, converging to the semiclassical measure $\mu_{sc}$ on $T^* M$. Suppose that $\{\psi_h\}_h$ are normalized quasimodes for $K_u(h)$ supported in the collar $B(0, \varepsilon_1)$ with central energy 0 and of width $\frac{c_2 h}{|\log h|}$, converging to a semiclassical measure $\sigma$ on $T^*(\mathbb{R}^r)$ such that $\sigma \geq \eta \delta_0$ (in other words, $\sigma$ gives mass at least $\eta$ to $\varepsilon_1$ in $(0, 1]$ to the point 0).

Then $\{\Psi_h\}_h$ are normalized quasimodes for $-\hbar^2 \Delta_N$ with central energies $E_h$ of width $\frac{(c_1 + c_2) h}{|\log h|}$, converging to a semiclassical measure giving mass at least $\eta$ to $\nu \mu_{sc}$ where $\nu: M \to N$ is the inclusion map.

In particular when $\eta = 1$ the quantum limit is exactly $\nu \mu_{sc}$.

**Proof.** Since our collar $N_{\varepsilon_1}(M)$ factors as a product $B(0, \varepsilon_1) \times M$ in the coordinate system, the identification of the limit is clear. It remains to verify that $\{\Psi_h\}_h$ are indeed quasimodes of the stated width. Following the calculation above we have:

$$(-\hbar^2 \Delta_N - E_h) \Psi_h = (K_u(h) \Phi_h)(u) \psi_h(x, \eta) + \Phi(u) \frac{1}{1 + |u|^2} (-\hbar^2 \Delta_M - E_h) \psi_h(x, \eta).$$

Now

$$\| (K_u(h) \Phi_h)(u) \|_{L^2(\mathbb{R}^r, (1 + u^2)^{-\frac{m-1}{2}} du)} \leq \frac{c_2 h}{|\log h|},$$
\[
\|\psi_h\|_{L^2(M)} = 1,
\]
\[
\left\| \frac{1}{1 + |u|^2} \Phi(u) \right\|_{L^2\left(\mathbb{R}^r, (1+u^2)^{m-1} du \right)} \leq \|\Phi(u)\|_{L^2\left(\mathbb{R}^r, (1+u^2)^{m-1} du \right)} = 1,
\]
and
\[
\left\| (-\hbar^2 \Delta_M - E_h) \psi_h \right\|_{L^2(M)} \leq \frac{c_1 \hbar}{|\log \hbar|},
\]
together giving the claim. \(\square\)

For the rest of the paper we will then construct transverse-to-\(M\) quasimodes \(\psi_h\) of central energy \(0\) for the operator \(K_u(h)\), such that the quasimodes microlocalize to the delta measure \(\delta_0(u)\) or at least contain this measure.

Writing \(\xi\) for the variable dual to \(u\) on \(\mathbb{R}^r\), the total symbol of \(K_u(h)\) as an \(h\)-pseudodifferential operator (see Section 3),
\[
|\xi|^2 + |u|^2 \left( \sum_{i,j} \frac{u_i u_j}{|u|^2} \xi_i \xi_j \right) + n \hbar (u \cdot \xi) - \frac{E_0 |u|^2}{1 + |u|^2} + f(h)
\]
hence giving a semiclassical principal symbol of
\[
\sigma(u; \xi) = |\xi|^2 + (u \cdot \xi)^2 - \frac{E_0 |u|^2}{1 + |u|^2};
\]
the reader is referred to [30] for further details on the symbol map. Observe that that \(\sigma\) differs from \(\tilde{\sigma} := |\xi|^2 - E_0 |u|^2\) (the Hamiltonian for the inverted harmonic oscillator) by terms of order \(O(|u| + |\xi|)^4\) when \((u; \xi)\) is sufficiently close to \((0; 0)\). It follows that that \(\sigma\) retains the non-degenerate critical point of \(\tilde{\sigma}\) at \((0; 0)\). The expression \(\tilde{\sigma}\) is a split quadratic form with eigenvalues \(\{\pm \mu_i\}_{i=1}^r\) where \(\mu_i > 0\). A linear change of variables along with \(\xi \to \sqrt{E_0} \xi\) transforms \(\tilde{\sigma}\) to \(\sum_{i=1}^r 2 \sqrt{E_0} \mu_i \xi_i\) whose Hamiltonian flow is the model for a hyperbolic fixed point.

We will obtain our quasimodes \(\psi_h\) from the following “quantized” version of a classical Birkhoff normal form (see the beautiful text [1] for more on this topic) specifically due to Iantchenko [19] but inspired by the work of Sjöstrand [25] in the context of Hamiltonians with non-degenerate minima. We state it in a slightly different fashion which is more adapted to our upcoming calculations:

**Proposition 4.8.** Let \(P(h)\) be a formally self-adjoint pseudodifferential operator (that is, a symmetric operator) acting on \(C^\infty_c(\mathbb{R}^r)\) (with a \(L^2(\mathbb{R}^r, \mu)\) structure for some positive measure \(\mu\)) with a real Weyl symbol \(\sum_{j=0}^\infty \hbar^j p_j\) whose semiclassical principal symbol \(p_0\) has a non-degenerate critical point and defines a split quadratic form, that is \(p(0; 0) = 0, dp(0; 0) = 0,\) and \(p''(0; 0)\) is non-degenerate but whose eigenvalues are \(\{\pm \lambda_i\}_{i=1}^r\) where \(\lambda_i > 0\). Set \(\lambda := (\lambda_1, \ldots, \lambda_r)\).

For \(N > 0\) given, we can find neighborhoods \(U, V\) of \((0; 0)\) in \(T^* \mathbb{R}^r\) and a canonical transformation \(\kappa : U \to V\), with \(\kappa(0; 0) = (0; 0)\), such that
\[
\left( \sum_{j=0}^N \hbar^j p_j \right) \circ \kappa = q^{(N)} + r_{N+1} = \sum_{j=0}^N \hbar^j q_j(x; \xi) + r_{N+1}(x; \xi; \hbar),
\]

where
\[ q_0 = \sum_{i=1}^{r} \lambda_i x_i \xi_i + \sum_{l=2}^{N} q_l^i \left( \sum_{\alpha-\beta \in \mathcal{M}_l, |\alpha|+|\beta|=l} c_{\alpha,\beta} x^\alpha \xi^\beta \right). \] (4.10)

Here \( \mathcal{M} = \{ \gamma \in \mathbb{N}^n : \vec{\lambda} \cdot \gamma = 0 \} \) is the so-called resonance module for the Hamiltonian \( q_{0,1} = \sum_i \lambda_i x_i \xi_i \) and \( r_{N+1}(x; \xi; \hbar) = \mathcal{O}\left((\hbar + |x| + |\xi|)^{N+1}\right) \). The other terms \( q_j \) have a similar form and for each \( j \), \( H_{q_0,1} q_j = 0 \).

Moreover, we have a corresponding quantum Birkhoff normal form. That is, there exists a microlocally unitary semiclassical Fourier integral operator \( U(h) : C^\infty_c(\mathbb{R}^r) \to S(\mathbb{R}^r) \) near \((0;0)\) (that is, with respect to the given \( L^2 \) structure, \( \|U(h)\|_{L^2} = \|\chi_2(x; hD_x)U(h)\chi_1(x; hD_x)\|_{L^2} \) for all microlocal cutoffs \( \chi_1, \chi_2 \) supported within \( U \) and \( V \), respectively) such that
\[ U_N^*(h)P(h)U_N(h) = Q^{(N)}(h) + R^{(N+1)}(h). \] (4.11)

Here, \( Q^{(N)}(h) \) and \( R^{(N+1)}(h) \) are semiclassical pseudodifferential operators with microlocal support in the indicated neighborhoods above with respective Weyl symbols \( q^{(N)} \) and \( r_{N+1} \) where \([\text{Op}_h^w(\sum_i \lambda_i x_i \xi_i), Q^{(N)}(h)] = 0\).

**Remark 4.12.** In our setting, the corresponding fixed point \((0;0)\) for our pre-normal form symbol in (4.7) yields \( \lambda_i = 2\sqrt{E_0} \) after the discussed scaling. For the sake of simplicity within our upcoming calculations and as the maximal Lyapunov exponent \( \lambda_E \) on the energy shell \( S^*_E N \) scales as \( \sqrt{E} \lambda_1 \), we consider the case of \( \sqrt{E} = 1 \) for the remainder of our article.

We note that as our submanifold \( M \) is arbitrary and considering the local coordinate expression in Lemma 2.6, our expansion rates transverse to \( M \) are equal across the entire ambient manifold \( N \) which is a feature of the constant curvature setting.

5. **Propagation of a Gaussian wavepacket at the Hyperbolic Fixed Point**

**Assumption 5.1.** For the remainder of this section until partway into Section 6, we keep the notation from Proposition 4.8 and substitute the use of \( u \in \mathbb{R}^r \) for \( x \in \mathbb{R}^r \). We will explicitly indicate later when we revert back to the use of \( u \) as our transversal variable to \( M \).

**Remark 5.2.** This Section gives the higher-dimensional analogues of the results in Section 5 of [13]. The arguments are very similar, but we give considerable detail for two reasons. First, while the dynamics of the \( r \)-dimensional inverted harmonic oscillator obviously factors into a product of \( r \) one-dimensional oscillators (for initial condition which factor), the quantum Birkhoff normal form includes additional error terms, and we need to track the dynamics after conjugating by an FIO. Second, the argument of Corollary 1.10 differs from previous quasimode constructions in that in it we vary the submanifold \( M \). Accordingly we must ensure that the spectral width of our quasimodes does not depend on the submanifold \( M \).

5.1. **Ground states, squeezed states, and evolution.**

5.1.1. **Preparing the Hamiltonian.** Consider the Gaussian
\[ \Phi_0(x) \overset{\text{def}}{=} \frac{1}{(\pi \hbar)^{r/4}} \prod_{i=1}^{r} \exp \left( -\frac{x_i^2}{2 \hbar} \right), \] (5.3)
which is the $L^2$-normalized ground state of the harmonic oscillator on $\mathbb{R}^r$ and has width $h^{1/2}$ in each direction $x_i$. We will evolve $\Phi_0$ through the reduced time-dependent Schrödinger equation generated by the QNF operator $Q^{(N)}$ described in Proposition 4.8, i.e. given

$$
\begin{align*}
\frac{i\hbar \partial_t \Phi_t^{(N)}}{D_t} &= Q^{(N)}(\hbar) \Phi_t^{(N)}, \quad \text{where } Q^{(N)} = Op^w_h(q^{(N)}) \\
\Phi_t^{(N)}|_{t=0} &= \Phi_0.
\end{align*}
$$

(5.4)

we must analyze the evolution of $\Phi_0(x)$. Therefore, our goal in this section is to describe the states

$$
\Phi_t^{(N)} = e^{-iQ^{(N)}t/\hbar} \Phi_0 \quad \text{for times } |t| \leq C(\tilde{\lambda}_1, \varepsilon_2) |\log \hbar|
$$

where $C > 0$ will be an explicit constant depending only on the maximal Lyapunov exponent $\tilde{\lambda}_1$ of $M$, in this case being 2, and a small parameter $\varepsilon_2$. The index $N$ will be chosen later.

The Weyl symbol of $Q^{(N)}$ arising from Proposition 4.8 can be rewritten as

$$
q^{(N)}(x, \xi; h) = (\sum_{i=1}^{r} q_{1,i}(h)x_i \xi_i) + \sum_{j=2}^{N} \left( \sum_{|\alpha|=|\beta|} q^{\alpha,\beta}_j(h)x^\alpha \xi^\beta \right),
$$

(5.5)

where the coefficients $q^{\alpha,\beta}_j(h)$ expand like

$$
q^{\alpha,\beta}_j(h) = \sum_{i=0}^{N} h^i q^{\alpha,\beta}_{j,i}, \quad \text{for } q^{\alpha,\beta}_{j,i} \text{ a constant}
$$

for $j \geq 2$. Moreover, for $j = 1$, $q_{1,i}(h) = \lambda_i + \mathcal{O}(h)$. Here, we have used that $\lambda_i = 2$ for all $i$ and thus that the resonance module is $\mathcal{M} = \{ \gamma \in \mathbb{N}^r : \sum_i \gamma_i = 0 \}$. This coefficient $q_{1,i}$ will play an important role in the Schrödinger evolution around the unstable fixed point $(0; 0)$.

In order to make the following analysis more transparent, we will truncate our Weyl symbol $q^{(N)}$ to its quadratic terms, by setting

$$
q^{(N)}_{\text{quad}} = \sum_{i=1}^{r} q_{1,i}(h)x_i \xi_i \quad \text{where } q_{1,i} = \lambda_i + \mathcal{O}(h).
$$

(5.6)

We label the operator whose Weyl symbol is $q^{(N)}_{\text{quad}}$ as $Q^{(N)}_{\text{quad}}(\hbar)$.

Recall from Proposition 4.8 that $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_r)$. The Schrödinger propagator for this special quadratic operator is easily expressed as

$$
U_{\text{quad}}(t) \Phi_0 := \exp \left( -\frac{it}{\hbar} Q^{(N)}_{\text{quad}} \right) \Phi_0 = D_{\tilde{\lambda}} \Phi_0,
$$

(5.7)

where the unitary dilation operator $D_{\tilde{\lambda}} : L^2(\mathbb{R}^r) \to L^2(\mathbb{R}^r)$ is given by

$$
D_{\tilde{\lambda}} u(x) := \exp \left( -i \sum_i \lambda_i \text{Op}^w_h(x_i \xi_i) / \hbar \right) u(x) = e^{-\sum_i \lambda_i/2} u(e^{-\lambda_1} x_1, \ldots, e^{-\lambda_r} x_r).
$$

(5.8)

The states $D_{\tilde{\lambda}} \Phi_0(x)$ and their generalizations are known as squeezed states. They naturally appear in related problems. Two applications include the Gutzwiller trace formula [11] and the pioneering work of Babich–Lazutkin [20] on the construction of quasimodes concentrating on closed elliptic geodesics. For more applications of these special states see [11] and the references therein.
For $N \geq 2$, the operator $Q^{(N)}$ includes a nonquadratic Hamilton $Q_{\text{quad}}^{(N)}$ with its Weyl symbol taking the form

$$q_{\text{quad}}^{(N)} \overset{\text{def}}{=} q^{(N)} - q^{(1)} = \sum_{j=2}^{N} \left( \sum_{|\alpha|=|\beta|=j} q_{j}^{\alpha,\beta}(h)x^{\alpha}z^{\beta} \right).$$

Unfortunately, these terms cannot be ignored and a large part of our construction will be in analyzing its contribution to the evolution of $\Phi_{0}$. However, the resonance condition given in Proposition 4.8 greatly simplifies the action of $Q_{\text{quad}}^{(N)}$ - it ensure that this operator commutes with $Q_{\text{quad}}^{(N)}$.

5.1.2. Squeezed excited states. We now recall some basic facts concerning the standard $r$-dimensional quantum harmonic oscillator $\sum_{i}(\hbar D_{x_{i}})^{2} + X_{i}^{2}$ where $X_{i}$ is the operator which multiplies by $x_{i}$. Our initial state $\Phi_{0} = \prod_{i=1}^{l} \varphi_{0}(x_{i})$ is also the ground state of this operator, where $\varphi_{0}(x_{i}) = \frac{1}{(\pi \hbar)^{1/4}} e^{-x_{i}^{2}/2\hbar}$ are themselves ground states of the 1-dimensional quantum harmonic oscillators $(\hbar D_{x_{i}})^{2} + X_{i}^{2}$.

Let us call $(\varphi_{m})_{m \geq 1}$ the 1-dimensional $m$-th excited states in the variable $x_{i}$, which are obtained by iteratively applying to $\varphi_{0}$ the “raising operator” $a_{i}^{*} \overset{\text{def}}{=} \text{Op}_{\hbar}(\frac{x_{i} + i\xi}{\sqrt{2\hbar}})$ and $L^{2}$-normalizing:

$$\varphi_{m} = \frac{(a^{*})^{m}}{\sqrt{m!}} \varphi_{0} \implies \varphi_{m}(x_{i}) = \frac{1}{(\pi \hbar)^{1/4} m^{1/2} \sqrt{m!}} H_{m}(x_{i}/\sqrt{\hbar}) e^{-x_{i}^{2}/2\hbar},$$

where $H_{m}(\cdot)$ is the $m$-th Hermite polynomial. We also have the dual lowering operators $a_{i} \overset{\text{def}}{=} \text{Op}_{\hbar}(\frac{x_{i} - i\xi}{\sqrt{2\hbar}})$ which satisfy the similar relation $a_{i} \varphi_{m} = \sqrt{m} \varphi_{m-1}$.

Now, given an $r$-vector of 1-dimensional excited states $(\varphi_{m_{1}}, \ldots, \varphi_{m_{r}})$, we can form the analogous $r$-dimensional excited state

$$\Phi_{m_{1}, \ldots, m_{r}}(x) \overset{\text{def}}{=} \prod_{i=1}^{r} \varphi_{m_{i}}(x_{i}).$$

This function continues to have $L^{2}$-norm 1, and by applying the unitary dilation operator $D_{\lambda}$ we obtain an $L^{2}$-normalized squeezed excited state $D_{\lambda}^{-1} \Phi_{m_{1}, \ldots, m_{r}}(x)$. These are essentially products of unitarily scaled Gaussians in one variable decorated by products of scaled polynomials. The main property exhibited by $D_{\lambda}^{-1} \Phi_{m_{1}, \ldots, m_{r}}$ that we will use is its similar concentration to $D_{\lambda}^{-1} \Phi_{0}$.

5.1.3. Expansion around the fixed point. The following result is inspired by the work of Combescure–Robert [11] and its proof is an $r$-dimensional version of Proposition 4.9 in [13]. Hence, we only provide the necessary details and leave the remaining elements to the reader.

**Proposition 5.11.** For every $l, N \in \mathbb{N}$, there exists a constant $C_{l,N} > 0$ and time dependent coefficients $c_{p}(t, h) \in \mathbb{C}$ such that the following estimate holds for any $h \in (0, 1]$: \n
$$\forall t \in \mathbb{R}, \quad \left\| e^{itQ^{(N)}/h} \Phi_{0} - D_{\lambda}^{-1} \Phi_{0} - \sum_{p=1}^{l} c_{p}(t, h) D_{\lambda}^{-1} \left( \sum_{k_{p}} d_{k_{p}} \Phi_{G(k_{p})} \right) \right\| \leq C_{l,N} \left( |t|h \right)^{l+1}.$$  

(5.12)
The coefficients \( c_p(t, h) \) are polynomials in \((t, h)\), of degree at most \(l\) in the variable \(t\). The \(d_k\) are constants, and \(G(p) \in \mathbb{N}^n\) index excited states. Furthermore, \(c_p(t, h) = \mathcal{O}(h^p)\) and the sums in \(k_p\) have finitely many terms.

**Remark 5.13.** This proposition will play a crucial role in determining the microlocal concentration of our future constructed quasimode. In order to show that the spectral width of our quasimodes is independent of the submanifold \(M\) we will need to keep track of the dependence of certain constants on the parameters \(l\) and \(N\). We will ultimately verify that any \(l \geq 2\) is sufficient for our purposes but that \(N\) may have to be large, specifically we shall require \((N + 1)\varepsilon_2/3 > 1\) where \(\varepsilon_2\) will appear in our Ehrenfest time.

**Proof.** As in [11], we would like to show that the full evolved state \(\Phi_t^{(N)} = U(t)\Phi_0 = e^{-iQ^{(N)}t/\hbar}\Phi_0\) is sufficiently approximated by \(U_{\text{quad}}(t)\Phi_0 = \mathcal{D}_x\phi_0(x)\), modulo a large sum of \(r\)-dimensional excited states whose \(L^2\)-norm is sufficiently small (bounded by an appropriate power of \(h\)).

The method of approximation is via the so-called Dyson expansion of Duhamel’s formula

\[
U(t) - U_{\text{quad}}(t) = \frac{1}{i\hbar} \int_0^t U(t - t_1)Q_{nq}U_{\text{quad}}(t_1) dt_1, \tag{5.14}
\]

which corresponds to the choice \(l = 1\). Using this formula directly leads to the bound

\[
\|U(t)\Phi_0 - U_{\text{quad}}(t)\Phi_0\| \leq Ct/\hbar \quad \text{for all } t \in \mathbb{R}, h \in [0, 1] \quad \text{(we will later have } t \leq C(\lambda_1, \varepsilon_2)|\log h|).\]

This isn’t sufficient since we want future error estimates to be \(\mathcal{O}(h^{-\delta})\) for \(\delta > 0\).

Accordingly we iterate the formula \(l > 1\) times to obtain

\[
U(t) - U_{\text{quad}}(t) = \sum_{j=1}^{l} \frac{1}{(i\hbar)^j} \int_0^t \int_{t_1}^t \cdots \int_{t_{j-1}}^t U_{\text{quad}}(t-t_j)Q_{nq}U_{\text{quad}}(t_j-t_{j-1})Q_{nq} \cdots Q_{nq} U_{\text{quad}}(t_1) dt_1 \cdots dt_j
\]

\[
+ \frac{1}{(i\hbar)^{l+1}} \int_0^t \int_{t_1}^t \cdots \int_{t_l}^t U(t - t_{l+1})Q_{nq}U_{\text{quad}}(t_{l+1} - t_l)Q_{nq} \cdots Q_{nq} U_{\text{quad}}(t_1) dt_1 \cdots dt_{l+1}.
\]

To simplify the notation, we shorten the last term to \(R_l^{(N)}(t, h)\).

A crucial fact involving the quantum Birkhoff normal form in Proposition 4.8 is that \([Q_{nq}^{(N)}, Q_{\text{quad}}^{(N)}] = 0\). This implies \(Q_{nq}^{(N)}\) also commutes with \(U_{\text{quad}}\), effectively giving us an exact Egorov formula when we apply quadratic evolution to resonant Hamiltonians: \((U_{\text{quad}})^*Q_{nq}^{(N)}U_{\text{quad}} = Q_{nq}^{(N)}\) since the Weyl symbol of \(Q_{nq}^{(N)}\) is a resonant function under the action of the Hamiltonian vector field of \(Q_{nq}^{(N)}\) (for proofs of the Egorov theorem in this case, see [11, 30]).

This commutativity leads us to the simpler expression

\[
U(t) - U_{\text{quad}}(t) = \sum_{p=1}^{l} \frac{t^p}{p!(i\hbar)^p} U_{\text{quad}}(t)(Q_{nq}^{(N)})^p + \frac{1}{(i\hbar)^{l+1}} \int_0^t \frac{t^l}{l!} U(t - t_{l+1})U_{\text{quad}}(t_{l+1})(Q_{nq}^{(N)})^{l+1} dt_{l+1} \tag{5.15}
\]

It is helpful to understand the explicit action of \(Q_{nq}^{(N)}\) on our initial state \(\Phi_0\). As

\[
((x + y)/2)^{\alpha} \xi^\beta = \prod_{i=1}^{r} \left( \sum_{\gamma_i} \left( \frac{\alpha_i}{\alpha_i - \gamma_i} \right) \frac{x_i}{2} \alpha_i - \gamma_i \xi_i^\beta \left( \frac{y_i}{2} \right)^{\gamma_i} \right)
\]
we see that

\[
\text{Op}_h^W(x^\alpha \xi^\beta) = \prod_{i=1}^r \left( \sum_{\gamma_i=0}^{\alpha_i} \left( \frac{\alpha_i}{\alpha_i - \gamma_i} \right) \left( \frac{x_i}{2} \right)^{\alpha_i - \gamma_i} (\hbar D_{x_i})^{\beta_i} \left( \frac{x_i}{2} \right)^{\gamma_i} \right). \tag{5.16}
\]

Using that \( x_i = \sqrt{\frac{\hbar}{2}} (a_i^* + a_i) \) and \( \hbar D_{x_i} = \sqrt{\frac{\hbar}{2}} (a_i^* - a_i) \), where \( a_i^* \) and \( a_i \) are the raising and lowering operators in the \( x_i \) variable, we find that (5.16) reduces to

\[
\text{Op}_h^W(x^\alpha \xi^\beta) = \prod_{i=1}^r \text{Op}_h^W(x_i^\alpha \xi_i^\beta) = \prod_{i=1}^r \left( \sum_{\gamma_i=0}^{\alpha_i} \left( \frac{\alpha_i}{\alpha_i - \gamma_i} \right) \left( \frac{1}{2} \right)^{\alpha_i} \left( \frac{\hbar}{2} \right)^{(\alpha_i + \beta_i)/2} (a_i^* + a_i)^{\alpha_i - \gamma_i} (a_i^* - a_i)^{\beta_i} (a_i^* + a_i)^{\gamma_i} \right). \tag{5.17}
\]

Here we have used composition formulae for the Weyl quantization and the Moyal product for the symbols in disjoint variables. Keeping in mind the action of the raising and lowering operators on \( \Phi_0 \), it follows that

\[
Q^{(N)}_{\text{inq}} \Phi_0 = \sum_{j=2}^{N} \sum_{|\alpha||\beta|=j} q_j^{\alpha,\beta}(\hbar) \text{Op}_h^W(x^\alpha \xi^\beta) \Phi_0 \tag{5.19}
\]

\[
= \sum_{j=2}^{N} \sum_{|\alpha||\beta|=j} q_j^{\alpha,\beta}(\hbar) O(h^{(|\alpha||\beta|)/2}) \left( \prod_{i=1}^r \sum_{k=0}^{\lceil \frac{\alpha_i + \beta_i + 1}{2} \rceil (\alpha_i + 1)} c_k^{\alpha_i,\beta_i} \varphi_k(x_i) \right) \tag{5.20}
\]

\[
= \sum_{j=2}^{N} O(h^2) \left( q_j^{\alpha,\beta}(\hbar) \prod_{i=1}^r \sum_{k=0}^{\lceil \frac{\alpha_i + \beta_i + 1}{2} \rceil (\alpha_i + 1)} c_k^{\alpha_i,\beta_i} \varphi_k(x_i) \right) \tag{5.21}
\]

\[
= \sum_{j=2}^{N} O(h^2) \left( q_j^{\alpha,\beta}(\hbar) \prod_{i=1}^r \sum_{k=0}^{\lceil \frac{\alpha_i + \beta_i + 1}{2} \rceil (\alpha_i + 1)} c(\alpha, \beta, k) \Phi_F(\alpha, \beta, k) \right) \tag{5.22}
\]

where \( P^j \) is the partition function and \( \Phi_F \) is an \( r \)-dimensional excited state with \( F \in \mathbb{N}^r \) being a function of \( \alpha, \beta, \) and \( k \). We have also used the fact that for odd (respectively, even) \( \alpha_i + \beta_i \), the operator \((a_i^* + a_i)^{\alpha_i - \gamma_i} (a_i^* - a_i)^{\beta_i} (a_i^* + a_i)^{\gamma_i} \) yields a sum of odd (respectively, even) indexed excited states when applied to \( \varphi_0(x_i) \). This is due to the fact that any word of length \( W \) consisting of \( a_i^* \) and \( a_i \) is a sum of words in “normal ordering” \((a_i)^m (a_i^*)^n\) (after using that \( [a_i^*, a_i] = 1 \)) each of whose length is equal in parity to \( W \). Hence, 

\[
(Q^{(N)}_{\text{inq}})^l \Phi_0 = \sum_{j=2l}^{Nl} O(h^j) \left( \sum_k d_k \Phi_{G(k)} \right) \tag{5.23}
\]

where \( d_k > 0 \) are constants, \( L(N, l) > 0 \) is a large (but computable) constant, and \( G(k) \in \mathbb{N}^n \). Moreover the \( L^2 \) norm of each term of (5.15) applied to our ground state, keeping in mind
that $U_{\text{quad}}(t)$ is unitary, is
\[
\left\| \frac{t^p}{p!(i\hbar)^p} U_{\text{quad}}(t) \left( Q_{nq}^{(N)} \right)^p \Phi_0 \right\| \leq C_{p,N} (|t|^p \hbar^p).
\] (5.24)

The constant $C_{p,N}$ grows exponentially in $N$ but since $N$ is independent of $\hbar$ this is not an issue.

We now return to the remainder term $R_t^{(N)}(t, \hbar)$ from (5.15). It yields
\[
\left\| \frac{1}{(i\hbar)^{l+1}} \int_0^t \frac{h^l}{l!} U(t - t_{l+1}) U_{\text{quad}}(t_{l+1}) (Q_{nq}^{(N)})^{l+1} \Phi_0 \ dt l+1 \right\| \leq C_{l,N} (|t|\hbar)^{l+1}.
\] (5.25)

Using these estimates, we group the terms (5.15) in increasing powers of $\hbar$. Although the notation is tedious, it useful to write the general expression of the $p$-th term in the sum (5.15) (removing the factor $\frac{t^p}{p!(i\hbar)^p}$):
\[
\sum_{j_1,\ldots,j_p} q_{j_1}^{(1)} \ldots q_{j_p}^{(p)}(\hbar) \ Op_h^w(x^{\alpha_1} \xi^{\beta_1}) \ldots Op_h^w(x^{\alpha_p} \xi^{\beta_p}) \Phi_0
\] (5.26)

Beyond the principal term $\mathcal{D}_{t,\tilde{\lambda}} \Phi_0$, the Dyson expansion then takes the form
\[
\mathcal{D}_{t,\tilde{\lambda}} \left[ \mathcal{O}_t(h) \left( \sum_{k_1} d_{k_1} \Phi_{G(k_1)} \right) + \mathcal{O}_t(h^2) \left( \sum_{k_2} d_{k_2} \Phi_{G(k_2)} \right) + \cdots + \mathcal{O}_t(h^l) \left( \sum_{k_l} d_{k_l} \Phi_{G(k_l)} \right) \right]
\] (5.27)
where the notation $\mathcal{O}_t(h^p)$ denotes a function $c_p(t, \hbar)$ which is at most $|t|^l$ and $\hbar^p$.

\[ \square \]

5.2. Microlocal support of the evolved state. Fix a small $\varepsilon_2 \in (0, 1)$, For $\hbar \in (0, 1/2]$ we define the local Ehrenfest time
\[
T_{\varepsilon_2} \overset{\text{def}}{=} \frac{(1 - \varepsilon_2)|\log \hbar|}{2\tilde{\lambda}},
\] (5.28)
where $\tilde{\lambda}$ is the maximal expansion rate amongst the individual rates $\{\lambda_i\}$; in our specific case, $\tilde{\lambda} = 2$.

**Proposition 5.29.** Let $\Phi_t^{(N)}$ be as in Proposition 5.11. Take $\Theta \in C^\infty(T^*({\mathbb{R}}^r))$ with $\Theta \equiv 1$ in a neighbourhood of $(0; 0)$, and denote its rescaling by $\Theta_\alpha(x, \xi) \overset{\text{def}}{=} \Theta(x/\alpha, \xi/\alpha)$. Then, for any power $M > 0$, normal form degree $N \in \mathbb{N}$ given in Proposition 4.8 and $\varepsilon_2 > 0$, there exists $C_{M,N} > 0$ such that
\[
\left\| \left[ \text{Op}_h(\Theta_{h^{2/3}}) - I \right] \Phi_t^{(N)} \right\|_{L^2} \leq C_{M,N} \hbar^M, \quad \hbar \in (0, 1/2],
\] (5.30)
uniformly for times $t \in [-T_{\varepsilon_2}, T_{\varepsilon_2}]$.

**Proof.** This is a variant of [13, Prop. 4.22]. However, again while our FIO will connect cutoffs to cutoffs it will not preserve the product structure, so we need to be careful.
We do begin with cutoffs given by a product, choosing \( \theta_i \in C^{\infty}_{c}([-2,2]^2) \) for each \( i = 1, \ldots, r \) such that \( \theta_i = 1 \) in \([-1,1]^2\), and scaling them as \( \theta_{i,n}(x,\xi) \overset{\text{def}}{=} \theta_i(x/\alpha,\xi/\alpha) \). It was shown in [13] that for any index \( m \) there exists \( C_m > 0 \) in a bounded range such that

\[
\| [\text{Op}_h^w(\theta_{i,\alpha}) - I]D_{\lambda_i} \|_{L^2} \leq C_m h^M,
\]

uniformly for \( |t| \leq T_{\varepsilon_2} \) and width \( \alpha \geq h^{\varepsilon_2/3} \).

To propagate the product we use Proposition 5.11 and the fact that each expression \( \sum_{k_p} d_{k_p} \Phi_{G(k_p)} \) is a sum of terms each of which is a product of scaled Gaussians \( \varphi_0 \) in disjoint variables (each of uniform width \( e^{\lambda_1 h^{1/2}} \leq h^{\varepsilon/2} \) by (5.28)) multiplied by polynomial factors. As we have a product structure in our evolved state given by Proposition 5.11, we have the same estimate for the individual terms:

\[
\left\| \left[ \text{Op}_h \left( \prod_{i=1}^r \theta_{i,\alpha} \right) - I \right] D_{\lambda} \right\|_{L^2} \leq C_{N,p} h^M.
\]

Now, consider a non-product form cutoff \( \Theta_\alpha \in C^{\infty}_{c}(\mathbb{R}^{2r}) \) equal to 1 on the support of \( \prod_{i=1}^r \theta_i \). Then

\[
\Theta_\alpha - 1 = (1 - \Theta_\alpha) \left( \prod_{i=1}^r \theta_{i,\alpha} - 1 \right).
\]

Taking \( \alpha \geq h^{\varepsilon_2/3}, \varepsilon_2/3 > 1/2 \) so that the semiclassical symbol class \( S^0_{\varepsilon_2/3} \) (see (3.5)) continues to have expansions in terms of increasing powers of \( h \), and using the support properties of \( \Theta_\alpha \) yields pseudodifferential cutoffs of the form

\[
\text{Op}_h^w (\Theta_\alpha - 1) = \text{Op}_h^w (1 - \Theta_\alpha) \text{Op}_h^w \left( \prod_{i=1}^r \theta_{i,\alpha} - 1 \right) + O_{L^2}(h^\infty).
\]

Our proof is complete after applying this operator to \( \Phi_t^{(N)} \) and using (5.32).

6. A log-scale quasimode for the Birkhoff normal form

A straightforward calculation gives \( \|Q^{(N)}(h)\Phi_0\| = (\prod_{i=1}^r \sqrt{2\lambda_i}) h + O_N(h^2) \) (recall that \( \Phi_0 \) is the standard Gaussian of equation (5.3)). In other words, applying the unitary Fourier integral operators of Proposition 4.8 to the quasimode constructed so far produces a quasimode \( \Psi_h \) for equation (4.1) with an unacceptable spectral width.

To get a narrower quasimode we use the usual time-averaging procedure originally due to Vergini–Schneider [27] and also employed in the predecessor works on scarring of quasimodes.
Let $T > 0$ be a parameter to be chosen later, fix a weight function $\chi \in C_c^\infty((-1, 1), [0, 1])$ and its rescaled version $\chi_T(t) \overset{\text{def}}{=} \chi(t/T)$. Our transverse quasimode will be

$$\Psi^{(N)}_{\chi_T, \hbar} \overset{\text{def}}{=} \int_{\mathbb{R}} \chi_T(t) e^{it(E_h - E_0)/\hbar} \Phi_t^{(N)} \, dt.$$  \hfill (6.1)

Note that this state is not yet normalized. In order to compute its spectral width, we will first need to compute its $L^2$ norm.

**Lemma 6.2.** For the semiclassically large averaging time $1 \leq T = T_h \leq C|\log \hbar|$ ($C > 0$ to be specified later) the square norm of out state $\Psi^{(N)}_{\chi_T, \hbar}$ satisfies

$$\|\Psi^{(N)}_{\chi_T, \hbar}\|^2 = T S(\lambda, f(h)/\hbar) \|\chi\|^2_{L^2} \left(1 + O_N(1/T)\right),$$

where $S(\bullet, \bullet)$ is a positive (and explicit) function $\hbar$ small enough, and $\lambda$ is the vector of expansion rates transverse to $M$ as seen in Proposition 4.8.

**Proof.** Although we obtained an explicit expression for the Dyson series (5.12), we prefer here the slightly less explicit operator equation (5.15) in order to reduce our calculations to those in the proof of [13, Lem. 5.4].

Begin with the representation of the evolved state as

$$\Phi_t^{(N)} = \left(U_{\text{quad}}(t) + \sum_{p=1}^l \frac{t^p}{p!(i\hbar)^p} U_{\text{quad}}(t)(Q_{nq}^{(N)})^p + R_t\right) \Phi_0$$ \hfill (6.3)

for some $l \in \mathbb{N}$ to be determined later. The norm squared of the averaged quasimode is then

$$\|\Psi^{(N)}_{\chi_T, \hbar}\|^2 = \int \langle \Phi^{(N)}_{t'}, \Phi^{(N)}_{t}\rangle \chi_T(t') \chi_T(t) \, dt \, dt'.$$ \hfill (6.4)

The key is then approximating the overlaps $\langle D_{t'\lambda}(Q_{nq}^{(N)})^p \Phi_0, D_{t'\lambda}(Q_{nq}^{(N)})^{p'} \Phi_0 \rangle = \langle (Q_{nq}^{(N)})^p \Phi_0, D_{t'\lambda}(Q_{nq}^{(N)})^{p'} \Phi_0 \rangle$.

Using again the resonance condition on the Weyl symbols we have that $(Q_{nq}^{(N)})^p$ is a power of a symmetric operator and that $(Q_{nq}^{(N)})^p$ commutes with $D_{t'\lambda}$. From these

$$\langle (Q_{nq}^{(N)})^p \Phi_0, D_{(t'-t)\lambda}(Q_{nq}^{(N)})^{p'} \Phi_0 \rangle = \langle (Q_{nq}^{(N)})^{p+p'} \Phi_0, D_{(t'-t)\lambda} \Phi_0 \rangle.$$ \hfill (6.5)

Since for our Euclidean quantization $Op_\hbar^w(x^\alpha \xi^\beta) = \prod_{i=1}^r Op_\hbar^w(x_i^\alpha \xi_i^\beta)$, the operator $(Q_{nq}^{(N)})^p$ maintains a product-type form into sums of differential operators in disjoint variables, it suffices to estimate products of the form

$$\prod_{i=1}^r \left(\sum_{m_i \geq 0} c_{m_i} \varphi_{m_i, D_{(t'-t)\lambda} \varphi_0} \right)_{\mathbb{R}^x_i},$$ \hfill (6.6)

where $K(N, p, p') > 0$.

A straightforward calculation shows

$$\langle \varphi_0, D_{(t'-t)\lambda} \varphi_0 \rangle_{\mathbb{R}^x_i} = \frac{1}{\sqrt{\cosh \lambda_i(t'-t)}}.$$

A similar differentiation identity for excited states developed in [13, Sec. 5.1] gives for each $m \in \mathbb{N}$ a constant $C_m > 0$ such that
\[
\left| \frac{\langle \varphi_m, \mathcal{D}(t'-t)\lambda, \varphi_0 \rangle_{\mathbb{R}^d_+}}{\langle \varphi_0, \mathcal{D}(t'-t)\lambda, \varphi_0 \rangle_{\mathbb{R}^d_+}} \right| \leq C_m \text{ uniformly in } t, t' \in \mathbb{R}. \tag{6.7}
\]
(in fact $C_m = 0$ for odd $m$ since in that case we are integrating an odd function against and even function).

Returning to equation (6.4), the term arising from the quadratic operator $U_{\text{quad}}(t)$ which corresponds to the case $m = 0$ takes the form
\[
I_0 = \int e^{itf(h)/\hbar} \left( \prod_{i=1}^r \langle \varphi_0, \mathcal{D}(t'-t)q_i^0 \varphi_0 \rangle_{\mathbb{R}^d_+} \right) \chi_T(t') \chi_T(t) dt \, dt'.
\]
This was evaluated in [13, Sec. 5.1], giving
\[
I_0(0) = T_s(\tilde{\lambda}, f(h)/\hbar) \|\chi\|_{L^2}^2 \left( 1 + \mathcal{O}(1/T) \right). \tag{6.10}
\]
It is also shown there that the correction terms arising from $m > 0$ (and the remainder $R_l$) are bounded above by $\tilde{C}_m h^T I_0(0)$ when $h$ is small enough. We have taken $T \sim |\log h|$ so the correction terms are $\mathcal{O}(h^\delta)$ for some $\delta > 0$ and are therefore lower order than the constant appearing in (6.10).

For this we need to bound $I_0(0)$, that is $S(\tilde{\lambda}, f(h)/\hbar)$, above and below. We would like to do this uniformly for $h$ small enough (this uniformity feeds into the argument of Corollary 1.10). As $f(h)/h = \mathcal{O}(1)$ and $S(\tilde{\lambda}, f(h)/h)$ is the 1-dimensional Fourier transform of a non-zero Schwarz function (for all $h$), the positivity of $\|\Psi^{(N)}_{\chi_T, E_h}\|_{L^2}$ establishes that of $S$.

We are can now define our penultimate normalized state
\[
\tilde{\Psi}^{(N)}_{\chi_T, h} \overset{\text{def}}{=} \frac{\psi^{(N)}_{\chi_T, h}}{\|\psi^{(N)}_{\chi_T, h}\|}. \tag{6.11}
\]

**Corollary 6.12.** Given $\varepsilon_2 > 0$ we may choose $T = T_{\varepsilon_2} \leq C(\varepsilon_2) |\log h|$ where $C(\varepsilon_2) > 0$ such that the normalized state $\tilde{\Psi}^{(N)}_{\chi_T, h}$ is localized in the $h^{2\varepsilon_2/3}$ neighbourhood of $(0; 0)$, in that for any $\Theta \in C_c^\infty(\mathbb{R})$ with $\Theta \equiv 1$ in a fixed neighbourhood of $(0; 0)$, we have the estimate
\[
\left\| (\text{Op}_h(\Theta_{h^{2\varepsilon_2/3}}) - I) \tilde{\Psi}^{(N)}_{\chi_T, h} \right\|_{L^2} = \mathcal{O}_N(h^{\infty}). \tag{6.13}
\]

**Proof.** This is immediate from Proposition 5.29 whilst keeping in mind that $T_{\varepsilon'}$ goes to the Ehrenfest time (5.28). \qed

**Proposition 6.14.** For $T = T_{\varepsilon_2}$, the spectral width of the state $\tilde{\Psi}^{(N)}_{\chi_T, h}$ at energy 0 is
\[
\left\| (Q^{(N)}(h) - f(h)) \tilde{\Psi}^{(N)}_{\chi_T, h} \right\|_{L^2}^2 = \frac{\hbar^2}{T_{\varepsilon_2}^2} \|\chi\|_{L^2}^2 \left( 1 + \mathcal{O}_N(1/|\log h|) \right). \tag{6.15}
\]
Hence, $\tilde{\Psi}_{\chi_T,h}$ gives an invariant semiclassical measure under the classical dynamics generated by $Q^{(N)}(h)$ and equal to $\delta_0(x) \in \mathcal{D}'(\mathbb{R}^r)$.

**Proof.** Recalling that $\Psi_{\chi_T,h}$ is defined by (6.1), we perform the following calculation:

$$
(Q^{(N)}(h) - f(h)) \Psi_{\chi_T,h} = \int_{\mathbb{R}} \chi_T(t) \left( Q^{(N)}(h) - f(h) \right) e^{-itQ^{(N)}(h) - f(h)/\hbar} \Phi_0 dt \\
= \int_{\mathbb{R}} \chi_T(t) i\hbar \partial_t \left( e^{-itQ^{(N)}(h) - f(h)/\hbar} \Phi_0 \right) dt \\
= -i\hbar \int_{\mathbb{R}} (\partial_t \chi_T(t)) e^{-itQ^{(N)}(h) - f(h)/\hbar} \Phi_0 dt.
$$

The norm of the last integral was essentially computed in Lemma 6.2 except our cutoff in time is now $-\hbar \chi'(t/T)/T$. A division by the asymptotic formula in the same Lemma for $\|\Psi_{\chi_T,h}\|$ finishes the first statement of our proposition.

The second statement follows from our quasimode having spectral width which is clearly $\alpha(h)$ and therefore yielding an invariant semiclassical measure under $\exp(tH_{\quad \text{quad}}^{(N)})$ [30]. The microlocal support statement from Lemma 6.12 tells us that this can only be $\delta_0$. \hfill \Box

The spectral width of the normalized mode $\tilde{\Psi}_{\chi_T,h}^{(N)}$ then takes the form

$$
F(h) = \frac{\hbar \|\chi'\|_{L^2}^2}{T_{\varepsilon_2} \|\chi\|_{L^2}^2} \left( 1 + \mathcal{O}_N(1/|\log \hbar|) \right),
$$

We note that the ratio $\frac{\|\chi'\|_{L^2}}{\|\chi\|_{L^2}}$ is essentially a universal constant. Since [13, Lem. 5.16] determined the infimum of these ratios to be $\pi/2$ we might as well make the minor improvement of choosing $\chi$ for which the ratio is $(1 + \varepsilon_2)\pi/2$.

The resulting spectral width is then

$$
F(h) = \pi \lambda_1 \frac{1 + \varepsilon_2}{1 - \varepsilon_2} \frac{h}{|\log \hbar|} + \mathcal{O}_N \left( \frac{h}{|\log \hbar|^\frac{3}{2}} \right) \\
\leq \pi \tilde{\lambda}_1 (1 + 3\varepsilon_2) \frac{h}{|\log \hbar|} + \mathcal{O}_N \left( \frac{h}{|\log \hbar|^\frac{3}{2}} \right).
$$

7. A Log-scale Quasimode on N and the Proof of Theorem 1.9

Having constructed quasimodes of spectral width $\frac{Ch}{|\log \hbar|}$ ($C$ depending on the fixed parameter $\varepsilon_2$ of the previous section) for our transverse dynamics in the Birkhoff Normal Form coordinates, we would like to transport them into our collar neighbourhood $N_{\varepsilon_1}(M)$ and plug them into our ansatz. The necessary ingredient is cutting off $\Psi_{\chi_T,h}$ in space.

We revert to our local Fermi-normal coordinate system $(x, \eta, u)$ in the collar neighbourhood $N_{\varepsilon_1}(M)$ determined in Section 2 where $(x, \eta)$ are local coordinates on $M$ and $u$ is an $r$-dimensional transverse variable. Our en energy level is now $E_0$ rather than 0.

By Lemma 4.5, it suffices to analyze the behaviour of the result of transporting the quasimode of the previous section to a quasimode for $K_u(h)$ in the $u$ variable, which we proceed to do.

Let $\Upsilon \in C_c^\infty(\mathbb{R}^r)$ be supported on $[-\varepsilon_1/2, \varepsilon_1/2]^r$ and equal to 1 on $[-\varepsilon_1/3, \varepsilon_1/3]^r$. 
Choose \( \varepsilon_2 > 0 \). We set \( \psi_h = \Upsilon(u)U_N \left( \tilde{\Psi}^{(N)}_{\chi_T,h} \right) \), that is

\[
\Psi_h = \varphi_h(x, \eta) \Upsilon(u)U_N \left( \tilde{\Psi}^{(N)}_{\chi_T,h} \right) (u). \tag{7.1}
\]

In the definition of \( \psi_h \), \( U_N(h) \) is the microlocally unitary FIO discussed in Proposition 4.8 and \( \tilde{\Psi}^{(N)}_{\chi_T,h} \) is as in equation (6.1). Then for \( h \) small enough, \( \psi_h \) will be supported in the range of the transverse variable \( u \) permitted in the collar neighbourhood making our function \( \Psi_h \) is well-defined.

We need to verify that the cutoff \( \Upsilon \) does not affect the norm, spectral behaviour, and concentration properties.

For the norm, Corollary 6.12 and properties of Fourier integral operators associated to canonical transformations (such as that in the statement of Proposition 4.8) tell us that for \( h \) small enough,

\[
\left\| \Upsilon U_N \left( \tilde{\Psi}^{(N)}_{\chi_T,h} \right) (u) \right\|_{L^2 \left( \mathbb{R}, (1+u^2)^{m-1/2} \, du \right)} = \left\| \tilde{\Psi}^{(N)}_{\chi_T,h} \right\| + \mathcal{O}_N(h^\infty) = 1 + \mathcal{O}_N(h^\infty). \tag{7.2}
\]

Note that the statement of Corollary 6.12 is exactly that \( \psi_h \) concentrate at 0 (that their Wigner measures converge weakly to the delta measure) in the Birkhoff normal form coordinates. Proposition 4.8 carries this to concentration near \( u = 0 \) for \( U_N \left( \tilde{\Psi}^{(N)}_{\chi_T,h} \right) \), and multiplying by the cutoff \( \Upsilon \) will not affect that.

For the spectral width, we need to verify that applying the FIO \( U_N \) and multiplication by \( \Upsilon \) have negligible effect. Concerning \( U_N \), we have

\[
\left\| K_u(h) \left( \Upsilon U_N \left( \tilde{\Psi}^{(N)}_{\chi_T,h} \right) \right) \right\|_{L^2 \left( \mathbb{R}, (1+u^2)^{m-1/2} \, du \right)} = \left\| U_N \left( (Q^{(N)}(h) + R_{N+1}(h)) \tilde{\Psi}^{(N)}_{\chi_T,h} \right) \right\| + \mathcal{O}_N(h^\infty). \tag{7.3}
\]

Proposition 6.14 along with our optimized cutoffs \( \chi \) then give:

\[
\left\| U_N \left( Q^{(N)}(h) \tilde{\Psi}^{(N)}_{\chi_T,h} \right) \right\| = \pi \tilde{\lambda}_1(1 + 3\varepsilon_2) \frac{h}{|\log h|} + \mathcal{O}_N \left( \frac{h}{|\log h|^2} \right). \tag{7.4}
\]

The contribution from the remainder \( R^{(N+1)}(h) \) is small: a second application of Corollary 6.12 tells us that \( R^{(N+1)}(h) \tilde{\Psi}^{(N)}_{\chi_T,h} = R^{(N+1)}(h) \Theta_{\varepsilon_2/3}(h) \tilde{\Psi}^{(N)}_{\chi_T,h} + \mathcal{O}_N(h^\infty) \). The composition calculus for Weyl quantizations gives \( R^{(N+1)}(h) \Theta_{\varepsilon_2/3}(h) = \text{Op}_h^N(r_{N+1}\Theta_{\varepsilon_2/3}) \). The Weyl symbol of this Moyal product satisfies the bound \( \mathcal{O}(h^{(N+1)\varepsilon_2/3}) \) after using the symbol estimates in Proposition 4.8.

Hence, letting \( (N+1)\varepsilon_2/3 > 1 \) and recalling that \( l \geq 1 \) from the Dyson expansion estimate (5.12) guarantees that our various remainder estimates are \( o(h/|\log h|^2) \).

Finally, multiplication by \( \Upsilon \) has no effect since the symbol of \( \Upsilon \) is the constant 1 near 0 the quasimode already concentrates near 0 before the multiplication.

Remark 7.5. We record here the order in which the various parameters in the construction have been chosen

- Given \( M \subseteq N \), there exists an \( \varepsilon_1 > 0 \) and a Collar neighborhood \( N_{\varepsilon_1}(M) \) in which we have a convenient coordinate system.
- Choose \( \varepsilon_2 > 0 \), giving the averaging time \( T_{\varepsilon_2} \).
• Choose \( N \) such that \( N + 1 > \frac{3}{\varepsilon_2} \).
• Choose \( l > 1 \), arbitrarily.
• Determine \( h_0 \) depending on all the previous choices such that if \( 0 < h < h_0 \) then the final state \( \Psi_h \) is microlocalized within the collar neighbourhood \( N_{\epsilon_1}(M) \).

Our parameters – which should be chosen in the order above – seem to depend on \( M \) only through its codimension \( r \). For example, we see no dependence on the expansion rates transverse to \( M \). This is an artefact of our hypothesis of constant negative curvature: the expansion rates are the same across all of \( N \).

REFERENCES


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