"COLLOCATION" FOR EIGENFUNCTIONS ON CONVEX POLYGONS

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Updated 12:40 with the L^∞ bound at the end.

Update 17:50 with a-prioi estimates

Update 2013/07/14, 00:45 with geometrical lemma and explicit bounds on Bessel functions.

Update 2013/07/15, 03:45 with more geometry in the sectors and derivative bounds.

1. Notation

- Fix a triangle $A_1A_2A_3$ in the plane, with angles parametrized by $\alpha_j = \frac{\pi}{\angle A_{j-1}A_jA_{j+1}}$. To the vertex A_j associate the sector S_j which is the intersection of the disc centered at A_j and tangent to $A_{j-1}A_{j+1}$ with the triangle.
- Let S_j be the sector obtained by intersecting the disc centered at A_j and tangent to $A_{j-1}A_{j+1}$ with the triangle.
  - Let R_j be its radius (the length of the altitude based at A_j).
  - In the sector let (r_j, θ_j) be the polar coordinate system based at A_j.
- Let S'_j be the subsector of radius $R'_j < R_j$.

Lemma 1. *Let O be the orthocentre of the triangle, and let $|A_jO| \leq R'_j \leq R_j$. Then the three sectors jointly cover the triangle.*

Proof. Let B_j be the foot of the altitude from A_j. Then the triangle A_jOB_{j+1} is right angled with hypotenuse A_jO, and in particular contained in any disc centered at A_j with radius at least $|A_jO|$. □

Lemma 2. *Let A_1, A_2 be points, let $R'_1, R'_2 > 0$ be real numbers such that $R'_1 + R'_2 > L = |A_1A_2|$, let O be a point at distance R'_i from A_i, B the projection of O on A_iA_{3-i} let $d_i = |BA_i|$. Let Ω be the intersection of the two circular arcs centered at A_i of radius R'_i bounded by A_iO and A_iA_{3-i}. For $0 \leq h \leq |OB|$ let I be the longest line segment perpendicular to BO at height h above B and contained in Ω.*

1. *The endpoints of I are*
2. *The length of I is*
3. *The area of Ω is*

Proof. A point of height h above A_1A_2 and at distance R'_i from A_i projects to the point at distance $\sqrt{(R'_i)^2 - h^2}$ on A_1A_2. The length of I is therefore

$$\sqrt{(R'_1)^2 - h^2} + \sqrt{(R'_2)^2 - h^2} - L$$

It also follows that $H = |BO|$ solves

$$\sqrt{(R_1')^2 - H^2} + \sqrt{(R_2')^2 - H^2} = L\,.$$

Squaring gives

$$2\sqrt{(R_1')^2 - H^2}\sqrt{(R_2')^2 - H^2} = L^2 + 2H^2 - \left((R_1')^2 + (R_2')^2\right)\,.$$

Squaring again gives

$$4\left((R_1')^2 - H^2\right)\left((R_2')^2 - H^2\right) = L^4 + 4H^4 + (R_1')^4 + (R_2')^4 + 4L^2H^2 - 2L^2\left((R_1')^2 + (R_2')^2\right) - 4H^2\left((R_1')^2 + (R_2'\right.$$

that is

$$4\,(R_1')^2\,(R_2')^2 = 4L^2H^2 + L^4 + \left((R_1')^2 + (R_2')^2\right)^2 - 2L^2\left((R_1')^2 + (R_2')^2\right)$$

and

$$H^2 = \frac{(R_1')^2\,(R_2')^2 - \frac{1}{4}\left[L^2 - \left((R_1')^2 + (R_2')^2\right)\right]^2}{L^2}$$

Now

$$\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

so the area is

$$\begin{aligned}\int_0^H \left[\sqrt{(R_1')^2 - h^2} + \sqrt{(R_2')^2 - h^2} - L\right]\mathrm{d}h &= \frac{H}{2}\left[\sqrt{(R_1')^2 - H^2} + \sqrt{(R_2')^2 - H^2}\right] + \frac{1}{2}\left[(R_1')^2 \arcsin\left(\frac{H}{R_1'}\right.\right. \\ &= -\frac{LH}{2} + \frac{1}{2}\left[(R_1')^2\,\angle OA_1A_2 + (R_2')^2\,\angle OA_2A_1\right] \\ &= \frac{1}{2}\left[(R_1')^2\left[\frac{\pi}{2} - \angle A_2\right] + (R_2')^2\left[\frac{\pi}{2} - \angle A_1\right]\right] - \frac{LH}{2ae}\end{aligned}$$

where $\angle A_1$ are the angles in the original triangle (vertex A_3 is a point such that A_3A_i is orthogonal to the line through A_{3-i}, O). □

2. A-priori esimates

Let u be an eigenfunction of Δ on a sector S of radius R, inverse angle α, and consider the restriction of u to the subsector of radius R'. On the big sector we have

$$u(r,\theta) = \sum_{k=0}^{\infty} a_k \tilde{J}_{k\alpha}\left(\sqrt{\lambda}r\right)\cos\left(k\alpha\theta\right)\,,$$

where we renormalize the Bessel function as $\tilde{J}_\alpha(z) = \Gamma(\alpha+1)J_\alpha(z)$ so that $\tilde{J}_\alpha(z) \sim \left(\frac{z}{2}\right)^\alpha$ for $0 < z \ll \sqrt{\alpha+1}$. Indeed, we have

$$\tilde{J}_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)}\left(\frac{z}{2}\right)^{2m}$$

and if $\alpha > -1$, $0 \le z \le 2\sqrt{\alpha+1}$ then this is an alternating zeries: the ratio of successive terms is $\left(\frac{z}{2}\right)^2 \frac{1}{(m+1)(\alpha+m+1)} \le \frac{\alpha+1}{\alpha+m+1}\frac{1}{m+1} \le 1$ so the series is alternating. In particular, in this range

$$\left(\frac{z}{2}\right)^\alpha\left[1 - \frac{\alpha+1}{\alpha+2}\right] \le \tilde{J}_\alpha(z) \le \left(\frac{z}{2}\right)^\alpha$$

that is

$$\frac{1}{\alpha+2}\left(\frac{z}{2}\right)^{\alpha} \leq \tilde{J}_{\alpha}(z) \leq \left(\frac{z}{2}\right)^{\alpha}$$

For a given r we have

$$\begin{aligned}\int_0^{\pi/\alpha_j} u(r,\theta)\cos(k\alpha\theta)\,d\theta &= \frac{1}{\alpha_j}\int_0^{\pi} u(r,\frac{\theta}{\alpha})\cos(k\theta)\,d\theta \\ &= \frac{\pi}{2\alpha}\tilde{J}_{k\alpha}\left(\sqrt{\lambda}r\right)\cdot a_k^j\,.\end{aligned}$$

It follows that

$$|a_k| \leq \frac{2\left\|u\right\|_{L^\infty(S)}}{\tilde{J}_{k\alpha}\left(\sqrt{\lambda}r\right)}\,.$$

If $2\sqrt{k\alpha_j+1} \geq \sqrt{\lambda}R$ then setting $r=R$ gives

$$|a_k| \leq \frac{2(k\alpha+2)\left\|u\right\|_\infty}{(\sqrt{\lambda}R/2)^{k\alpha}}\,.$$

Thus on S'_j we have the a-priori bound

$$\begin{aligned}\left|\sum_{k=K}^{\infty} a_k\tilde{J}_{k\alpha}\left(\sqrt{\lambda}r\right)\cos(k\alpha\theta)\right| &\leq 2\left\|u\right\|_\infty \sum_{k=K}^{\infty}(k\alpha+2)\left(\frac{R'}{R}\right)^{k\alpha} \\ &= \left\|u\right\|_\infty\left(\frac{R'}{R}\right)^{K\alpha}\left[\frac{K\alpha+2}{1-\left(\frac{R'}{R}\right)^\alpha}+\alpha\frac{1}{\left[1-\left(\frac{R'}{R}\right)^\alpha\right]^2}\right].\end{aligned}$$

In particular, we can make the truncation error less than ϵ by taking K large enough. Note that this requires an a-priori bound on $\left\|u\right\|_\infty$ which I think is available.
Gradient estimates. Let's start with the easy case. For derivative wrt θ we need to bound (the factor r comes from the metric in polar coordinates)

$$\begin{aligned}r\left|-\sum_{k=K}^{\infty} a_k\tilde{J}_{k\alpha}\left(\sqrt{\lambda}r\right)k\alpha\sin(k\alpha\theta)\right| &\leq 2R'\alpha\left\|u\right\|_\infty \sum_{k=K}^{\infty}k\,(k\alpha+2)\left(\frac{R'}{R}\right)^{k\alpha} \\ &= \left\|u\right\|_\infty\left(\frac{R'}{R}\right)^{K\alpha}R'\alpha\left[\frac{K(K\alpha+2)}{1-\left(\frac{R'}{R}\right)^\alpha}+\frac{2K\alpha+2}{\left[1-\left(\frac{R'}{R}\right)^\alpha\right]^2}+\frac{2\alpha}{\left[1-\left(\frac{R'}{R}\right)^\alpha\right]^3}\right.\end{aligned}$$

For the derivative wrt r we need to bound

$$\left|\sum_{k=K}^{\infty} a_k\tilde{J}'_{k\alpha}\left(\sqrt{\lambda}r\right)\cos(k\alpha\theta)\right|\,.$$

Now the ratio of successive terms in the series for J' is

$$\leq \frac{\alpha+1}{\alpha+m+1}\frac{1}{m+1}\frac{2m+2+\alpha}{2m+\alpha} = \frac{1}{m+1}\frac{2m\alpha+\alpha^2+2m+3\alpha+2}{2m\alpha+\alpha^2+2m+(m+1)\alpha+2m^2} < 1$$

for $m \geq 2$. It follows that

$$J'_\alpha(z) \leq \frac{\alpha}{2}\left(\frac{z}{2}\right)^{\alpha-1} - \frac{\alpha+2}{2(\alpha+1)}\left(\frac{z}{2}\right)^{\alpha+1} + \frac{\alpha+4}{2(\alpha+1)(\alpha+2)}\left(\frac{z}{2}\right)^{\alpha+3}$$

and for $z \le 2\sqrt{\alpha+1}$ this reads

$$\begin{aligned} |J'_\alpha(z)| &\le \frac{\alpha}{2}\left(\frac{z}{2}\right)^{\alpha-1}\left[1+\frac{(\alpha+4)(\alpha+1)}{\alpha(\alpha+2)}\right] \\ &\le \frac{\alpha}{2}\left(\frac{z}{2}\right)^{\alpha-1}\left[1+1+\frac{3}{\alpha+2}+\frac{4}{\alpha(\alpha+2)}\right] \\ &\le 3\alpha\left(\frac{z}{2}\right)^{\alpha-1} \end{aligned}$$

since $\alpha \ge 2$ in an acute angle. Thus

$$\begin{aligned} \left|\sum_{k=K}^{\infty} a_k \tilde{J}'_{k\alpha}\left(\sqrt{\lambda}r\right)\cos\left(k\alpha\theta\right)\right| &\le \sum_{k=K}^{\infty}\frac{2(k\alpha+2)\left\|u\right\|_\infty}{(\sqrt{\lambda}R/2)^{k\alpha}}3\alpha\left(\frac{\sqrt{\lambda}R'}{2}\right)^{k\alpha-1} \\ &\le \frac{6}{\sqrt{\lambda}R'}\left\|u\right\|_\infty\left(\frac{R'}{R}\right)^{K\alpha}\left[\frac{K\alpha+2}{1-\left(\frac{R'}{R}\right)^\alpha}+\alpha\frac{1}{\left[1-\left(\frac{R'}{R}\right)^\alpha\right]^2}\right] \end{aligned}$$

which should be fine given some lower bound on $\sqrt{\lambda}$.
The case of small angles.

Problem 3. Is the ratio $\left(\frac{R'}{R}\right)^\alpha$ uniformly bounded away from 1? [if R' is close to R then the angle at A is small, so α is large]

3. Numerical scheme

- Fix a guess λ for an eigenvalue.
- Fix parameters K, M.
- Let $\left\{a_k^j\right\}_{k=0,j=1}^{k=K-1,j=3}$ be unknowns (variables) except that $a_0^1 = 1$ (pinned), and let $\underline{x}$ be the vector of the unknowns (all a_k^j except a_0^1).
- Choose m points z_i in the triangle such that each z_i lies in at least two sectors. For each i,j let $r_i^j = |z_i - A_j|$ and let θ_i^j be the angle the vector $\overline{A_j z_j}$ makes with the side $A_j A_{j-1}$.
- Let $T \in M_{3M,3K-1}(\mathbb{R})$ and $\underline{b} \in \mathbb{R}^M$ be the following matrix and vector: for each $0 \le i \le M-1$, choose two vertices (say A_j, A_{j+1} such) that $z_i \in S_j \cap S_{j+1}$, and consider the equivalent statements:

$$\sum_{k=0}^{K} a_k^j J_{k\alpha_j}\left(\sqrt{\lambda}r_i^j\right)\cos\left(k\alpha_j\theta_i^j\right) = \sum_{k=0}^{K} a_k^{j+1} J_{k\alpha_{j+1}}\left(\sqrt{\lambda}r_i^{j+1}\right)\cos\left(k\alpha_{j+1}\theta_i^{j+1}\right).$$

$$\sum_{k=0}^{K} a_k^j J_{k\alpha_j}\left(\sqrt{\lambda}r_i^j\right)\cos\left(k\alpha_j\theta_i^j\right) - \sum_{k=0}^{K} a_k^{j+1} J_{k\alpha_{j+1}}\left(\sqrt{\lambda}r_i^{j+1}\right)\cos\left(k\alpha_{j+1}\theta_i^{j+1}\right) = 0$$

  Separating out the term a_0^1 if it appears and shifting it to the RHS gives the inner product of a vector with $\underline{x}$, and we let that vector be the $3i$th row of of T and let b_{3i} be the coefficient of a_0^1 if it appears (zero otherwise).
  - Similarly, write down the equations stating that the gradients of the two functions agree at z_i. Since the constant terms do not contribute we set $b_{3i+1} = b_{3i+2} = 0$.
- Find $\underline{x}$ (depending on λ) minimizing $R = \left\|A\underline{x} - \underline{b}\right\|^2$.

- Plot the distance $R(\lambda)$ as a function of λ. If it dips sharply we found a suspected eigenfunction.

4. Terry's Post-solution step: How to combine the three expansions into a single function

Choose a smooth partition of unity $1 = \sum_{j=1}^{3} \psi_j$ of the triangle such that ψ_j is supported in the sector S_j and such that the normal derivative of each ψ_j at the boundary of the triangle is identically zero. Note that we can choose ψ_j in advance.

Now let $u_j(z) = \sum_{k=0}^{K} a_j^k J_{k\alpha_j}\left(\sqrt{\lambda} r_i^j\right) \cos\left(k\alpha_i \theta_i^j\right)$ be the functions with a_k^j as computed numerically, Set

$$u(z) = \sum_{j=1}^{3} \psi_j(z) u_j(z)\,.$$

Lemma 4. *$u(z)$ is smooth in the triangle and satisfies the Neumann boundary condition.*

Proof. Clearly smooth. For z on the boundary,$\partial_{\mathrm{n}} u(z) = \sum_{j=1}^{3} \left[\psi_j(z)\partial_{\mathrm{n}} u_j(z) + u_j(z)\partial_n \psi_j(z)\right] = 0$ since $\partial_n \psi_j(z) = 0$ by its choice and $\partial_{\mathrm{n}} u_j(z) = 0$ on $\partial\left(A_1 A_2 A_3\right) \cap \operatorname{supp}(\psi_j)$ by the functional form of u_j. □

Lemma 5. *$\|(\Delta - \lambda)u\|$ is small.*

Proof. Set

$$\begin{aligned} E(z) = (\Delta - \lambda)\, u(z) &= \sum_{j=1}^{3} \left[\left(\psi_j\left(\Delta - \lambda\right) u_j\right) + u_j \Delta\psi_j + 2\nabla u_j \nabla\psi_j\right] \\ &= \sum_{j=1}^{3} \left[u_j \Delta\psi_j + 2\nabla u_j \nabla\psi_j\right]\,. \end{aligned}$$

Now at any point $z \in A_1A_2A_3$, if z belongs to a unique sector then $\psi_j \equiv 1$ near z and the expression vanishes. Otherwise, let z_i be the nearest point among the originally prescribed points. Then can bound $E(z) - E(z_i)$ by derivative estimates on ψ_j (which is fixed) and on u_j (numerically, given the coefficients). Clearly $E(z_i)$ is closely related to $(A\underline{\mathrm{x}} - \underline{\mathrm{b}})_{3i,3i+1,3i+2}$: in the sum up to this error we can replace all $u_j, \nabla u_j$ with $u_1, \nabla u_1$ (wlog $z \in S_1$) and then all derivatives of $\sum_{j=1}^{3} \psi_j \equiv 1$ vanish.

Finally, assuming the z_i are well-distributed, $\int |E(z)|^2 \,\mathrm{d}z$ should be close to $\|A\underline{\mathrm{x}} - \underline{\mathrm{b}}\|_2^2 = R$, so we have an estimate on $\|(\Delta - \lambda)u\|_{L^2(A_1A_2A_3)}$. □

Corollary 6. *There is an eigenvalue of the triangle close to λ.*

5. B–S–V (idea, I haven't actually tried to translate it to the current setting)

Given the "guess" vector $\underline{\mathrm{x}}$, do a further (reasonable) computation which, if successful (always in practice, doesn't have to in theory), shows that for some reasonably large $K_0 \le K$ the coefficients $\left\{a_k^j\right\}_{k \le K_0}$ are ϵ-accurate.

Corollary 7. *L^∞ bounds on the distance between the numerical solution and the true eigenfunction.*

Proof. Make a Bessel expansion as before, and truncate at K_0. The error due to truncation at K_0 can be bounded a-priori, while for the a_k^j where $k \leq K_0$ we have an explicit bound on their distance from the expansion of the true solution. □