Math 538, Lecture 9, 7/2/2024

Last time: $\mathbb{Q}_p$: completion of $\mathbb{Q}$ at $|\cdot|_p$

$$\mathbb{Q}_p = \left\{ \sum_{i=m}^{\infty} a_i p^i \;\middle|\; \begin{array}{l} a_i \in \{0, \ldots, p-1\} \\ m \in \mathbb{Z}, a_m \neq 0 \end{array} \right\}$$

$$v_p\left(\sum_{i=m}^{\infty} a_i p^i\right) = m$$

Ex: Define $+, \cdot$ via addition with carry, Cauchy prod, check properties

identification of $\mathbb{Z}_p$ with $[p]^{\mathbb{N}}$ is a homeo
$\Rightarrow \mathbb{Z}_p$ is cpt, $\mathbb{Q}_p$ locally compact.

Also saw: ideals of $\mathbb{Z}_p$ are exactly $p^k\mathbb{Z}_p$, $k \geq 0$,

$$\mathbb{Z}_p / p^k\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$$

HW: $\mathbb{Z}_p$ also the inverse limit of $\mathbb{Z}/p^k\mathbb{Z}$.

Today: Complete fields

Fix field F, complete wrt a non-discrete absolute value $|\cdot|$.

__Def:__ A topological (F-)vector space is an F-vsp V equipped with a Hausdorff topology st. the maps

$$V \times V \to V \qquad F \times V \to V$$
$$(\underline{u}, \underline{v}) \mapsto \underline{u} - \underline{v} \qquad (\alpha, \underline{v}) \mapsto \alpha \underline{v}$$

are cts

__Example:__ F^n with pdt topology (or with $|v|_\infty = \max\{|v_i|\}_{i=1}^n$)

__Observation:__ $\mathrm{Hom}_F(F^n, F^m) = \mathrm{Hom}_{TVS}(F^n, F^m)$

__Thm:__ (1) Any f.d. TVS is linearly homeomorphic to F^n. (2) Any f.d. subspace of a TVS is closed.

Fix TVS V.

__Lemma:__ Let $\underline{0} \in \mathcal{U} \subset V$ be a nbd of $\underline{0}$. Then there is open $\underline{0} \in \mathcal{U}' \subset \mathcal{U}$ st. $x\mathcal{U}' \subset \mathcal{U}'$ if $|x| \leq 1$ (say $\mathcal{U}'$ is balanced)

Pf: The set $\{(x, \underline{v}) \mid x\underline{v} \in \mathcal{U}\} \subset F \times V$ is open $\Rightarrow$ contains subset of the form $B(0,r) \times \mathcal{U}_1$, $\mathcal{U}_1$ nbd of $\underline{0}$. Take $x \in F$ st $|x| > \frac{1}{r}$ (use $|\cdot|$ is nondiscrete)

Then $x \cdot B(0,r) = B(0, r|x|) \supseteq B(0,1)$,

$\mathcal{U}_2 = x^{-1}\mathcal{U}_1$ is open,

$$\mathcal{U}' = \underset{\text{open}}{B_F(0,1)\,\mathcal{U}_2} \subset x B_F(0,r)\, x^{-1}\mathcal{U}_1 \subset \mathcal{U}$$

Lemma: A complete subspace $W \subset V$ is closed in V.

Pf: Let $\{w_i\}_{i \in I}$ be a net in W converging to $\underline{v} \in V$.
Then $\{w_i\}_{i \in I}$ is Cauchy $\Rightarrow$ convergent in W (complete)
$\Rightarrow \underline{v} \in W$ (Hausdorff).

Prop: Let V be an F-TVS. Then every 1d subspace $W \subset V$ is linearly homeomorphic to F
$\Rightarrow$ complete $\Rightarrow$ closed.

Pf: Let $\underline{w} \in V$ be non-zero st $W = F \cdot \underline{w}$.
Have $f: F \to W$ $\quad f(x) = x\underline{w}$.

Clearly cts, bijective. Want f to be open.
Since V is Hausdorff have open nbd $\underline{0} \in \mathcal{U} \not\ni \underline{w}$

Wlog $\mathcal{U}$ is balanced. Then $\{x \in F \mid x\underline{w} \in \mathcal{U}\} = f^{-1}(\mathcal{U})$ is non-empty (contains 0), open (continuity f), in't by $B(0,1)$, does not contain 1, so contained in $B(0,1)$

$\Rightarrow \quad f(B(0,1)) \supset \mathcal{U} \cap W \leftarrow$ nbd of $\{0\}$ in W.

$\Rightarrow$ (translation, rescaling) f is open. ∎

<u>Pf of thm</u>: By induction on $\dim_F V$.

Suppose $\dim_F V = n+1$, thm known if $\dim_F V = n$

Fix basis $\{\underline{v}_i\}_{i=1}^{n+1} \subset V$, set $W_1 = \operatorname{Span}\{\underline{v}_i\}_{i=1}^{n}$, $W_2 = F\cdot \underline{v}_{n+1}$.

By induction $W_1 \cong F^n$, $W_2 \cong F$, so $W_1, W_2 \subset V$ are complete, so closed.

As before $f: F^{n+1} \to V$ $f(\underline{\alpha}) = \sum_{i=1}^{n+1} \alpha_i \underline{v}_i$ is linear isom, cts, want cts inverse.

Note: since W_i closed, V/W_i are Hausdorff.

linear isom

$$V \to (V/W_1) \times (V/W_2) \quad \text{cts}$$

By induction: $\qquad \overset{\cong}{F} \qquad \times \qquad \overset{\cong}{F^n}$

compose with automorphism of $F \times F^n \cong F^{n+1}$ to set to inverse of f.

If W is any F-TVS, $V \subset W$ f.d. then $V \cong F^n \Rightarrow V$ complete $\Rightarrow V$ closed. ∎

<u>Cor</u>: Let L/F be an algebraic extension. Then there is at most one extension of $|\cdot|_F$ to L.

<u>Pf</u>: Any such extension gives L the structure of a TVS $/F$. Let $x \in L$. Since x is alg. $/F$, $F(x) \subset L$ is f.d. so its topology is unique, so equivalence class of $|\cdot|_2 \restriction_{F(x)}$ is determined.

If $|\cdot|_1, |\cdot|_2$ are two extensions to $F(x)$, have $|\cdot|_1 = |\cdot|_2^{\lambda}$, but must have $\lambda = 1$ since the two agree on F. ∎

<u>Cor</u>: Let L/F be a finite extension of fields, $|\cdot|_w$ an absolute of L s.t. $|\cdot|_v = |\cdot|_w \restriction_F$ is non-trivial. Then L_w is an algebraic extension of F_v, in fact $[L_w : F_v] \leq [L : F]$.

Pf: The subspace $L \cdot F_v \subset L_w$ is f.d
$= \mathrm{Span}_{F_v}\{F\text{-basis of } L\}$

$\Rightarrow$ closed, contains dense subset L

so any F-basis of L spans L_w over F_v. $\square$

Continue with F complete wrt $|\cdot|$, assume $|\cdot|$ is nonarchimedean. Goal: Extend $|\cdot|$ to algebraic extensions.

Lemma; let $\mathcal{O} = \{x \in F : |x| \leq 1\}$
$\mathfrak{p} = \{x \in F : |x| < 1\}$

Then:

(1) $\mathcal{O}$ is a subring of F, the maximal bounded subring.

(2) K is the field of fractions of $\mathcal{O}$, $\mathcal{O}$ integrally closed in K

(3) $\mathfrak{p}$ is an ideal of $\mathcal{O}$, the unique maximal ideal.

(4) $\mathcal{O}^\times = \{x \in F : |x| = 1\}$

Pf: HW

Cor: $\mathbb{Z}$ integrally closed in $\mathbb{Q}$.

Pf: Let $\alpha \in \mathbb{Q}$ s.t. $f(\alpha)=0$ where $f \in \mathbb{Z}[x]$ monic. Then for any p, $f(\alpha)=0$ if we view $f \in \mathbb{Z}_p[x]$ so $\alpha \in \mathbb{Z}_p$, i.e. $v_p(\alpha) \geq 0$, denominator of α not divisible by p. But p was arbitrary so $\alpha \in \mathbb{Z}$

Notation: $k = \mathcal{O}/\mathfrak{p}$ is the residue field

For $\alpha \in \mathcal{O}$ write $\bar{\alpha}$ for image in k.

For $f \in F[x]$, say $f = \sum_{i=0}^{d} a_i x^i$ write

$$|f| = \max\{|a_i|\}_{i=0}^{d} \qquad (\text{"content" of } f)$$

Call $f \in \mathcal{O}[x]$ primitive iff $|f|=1 \iff \bar{f} \neq 0 \in k[x]$

Prop: (Hensel's Lemma). Let $f \in \mathcal{O}[x]$

(1) Suppose that for some $\alpha \in \mathcal{O}$, $|f(\alpha)| < |f'(\alpha)|^2$.

Then there is $\beta \in \mathcal{O}$ s.t. $f(\beta)=0$, and in fact

$$|\alpha - \beta| \leq \left|\frac{f(\alpha)}{(f'(\alpha))^2}\right| < 1.$$

(2) Suppose $\bar{f} \neq 0$, and that $\bar{f} = \bar{g}\bar{h}$ in $k[x]$ where $\bar{g}, \bar{h}$ are relatively prime. Then $\exists g, h \in \mathcal{O}[x]$ lifting $\bar{g}, \bar{h}$ s.t. $f = gh$, $\deg g = \deg \bar{g}$.

Pf: If $f \in R[x]$, then $f(x) - f(y) = (x-y)\cdot g(x,y)$ for some $g \in R[x,y]$ $\left(x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1})\right)$

$\Rightarrow$ if $f \in \mathcal{O}[x]$, $\alpha, \beta \in \mathcal{O}$ then $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$

If $|\alpha - \beta| < |f'(\alpha)|$ then applying claim to f', get

$$|f'(\alpha) - f'(\beta)| \leq |\alpha - \beta| < |f'(\alpha)|$$

so $|f'(\beta)| = |f'(\alpha)| > 0$.

Also, in $f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + g(x)(x-\alpha)^2$

since $f'(\alpha) \in \mathcal{O}$, $g \in \mathcal{O}[x]$.

Now set $c = \left|\dfrac{f(\alpha)}{(f'(\alpha))^2}\right| < 1$, define $\alpha_0 = \alpha$,

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

Suppose by induction $|f'(\alpha_n)| = |f'(\alpha)|$, $\left|\frac{f(\alpha_n)}{(f'(\alpha_n))^2}\right| \le C^{2^n}$, $\alpha_n \in \mathcal{O}$

$$\text{so } \left|\frac{f(\alpha_n)}{(f'(\alpha_n))}\right| = \left|\frac{f(\alpha_n)}{(f'(\alpha_n))^2}\right| \cdot |f'(\alpha_n)| \le 1 \cdot 1 = 1$$

so $\frac{f(\alpha_n)}{f'(\alpha_n)} \in \mathcal{O}$ so $\alpha_{n+1} \in \mathcal{O}$ $\quad (|\alpha_{n+1} - \alpha_n| \le C^{2^n})$

As before $|f'(\alpha_{n+1}) - f'(\alpha_n)| \le |\alpha_{n+1} - \alpha_n|$

$$= \left|\frac{f(\alpha_n)}{f'(\alpha_n)}\right| = \left|\frac{f(\alpha_n)}{(f'(\alpha_n))^2}\right| \cdot |f'(\alpha_n)|$$

$$< |f'(\alpha_n)|$$

$$\Rightarrow |f'(\alpha_{n+1})| = |f'(\alpha_n)| = |f'(\alpha)|$$

Finally,

$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$ $\qquad \gamma = g(\alpha_n) \in \mathcal{O}$

$$\frac{f(\alpha_{n+1})}{(f'(\alpha_{n+1}))^2} = \frac{f(\alpha_n) + f'(\alpha_n)(\alpha_{n+1} - \alpha_n) + \gamma(\alpha_{n+1} - \alpha_n)^2}{(f'(\alpha_{n+1}))^2}$$

$$= \frac{\gamma(\alpha_{n+1} - \alpha_n)^2}{(f'(\alpha_{n+1}))^2}$$

$$\Rightarrow \left|\frac{f(\alpha_{n+1})}{(f'(\alpha_{n+1}))^2}\right| \le \left|\frac{\alpha_{n+1} - \alpha_n}{(f'(\alpha_n))}\right|^2 \le \left|\frac{f(\alpha_n)}{(f'(\alpha_n))^2}\right|^2 \le (C^{2^n})^2.$$

$\Rightarrow \alpha_{n+1} - \alpha_n \to 0 \Rightarrow \sum_{n=0}^{\infty} (\alpha_{n+1} - \alpha_n)$ converges

i.e. $\beta = \lim_{n\to\infty} \alpha_n$ exists

since $|\alpha_{n+1} - \alpha_n| \leq C^{2^n}$, $|\alpha_n - \alpha_0| \leq C$

so $|\beta - \alpha| \leq C$; clearly $f(\beta) = 0$

since $|f(\alpha_n)| \leq C^{2^n} \to 0$