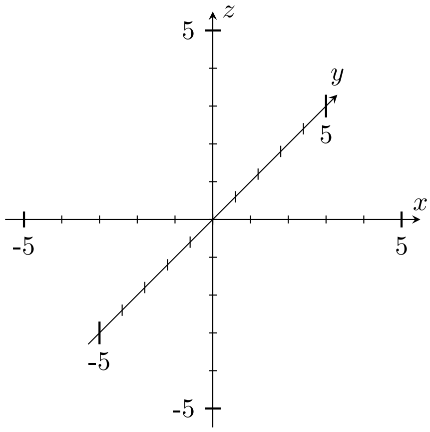
Math 100C – SOLUTIONS TO WORKSHEET 10
MULTIVARIABLE CALCULUS

1. Plotting in three dimensions

(1) $\star$ Plot the points $(2,1,3)$, $(-2,2,2)$ on the axes provided.



(2) Let $f(x,y) = e^{x^2+y^2}$.

(a) $\star$ What are $f(0,-1)$? $f(1,2)$? Plot the point $(0,1,f(0,1))$ on the axes provided.

(b) $\star$ What is the *domain* of f (that is: for what (x,y) values does f make sense?

Solution: f makes sense for all (x,y) – equivalenly that is on the plane $\mathbb{R}^2$.

(c) $\star$ What is the *range* of f (that is: what values does it take)?

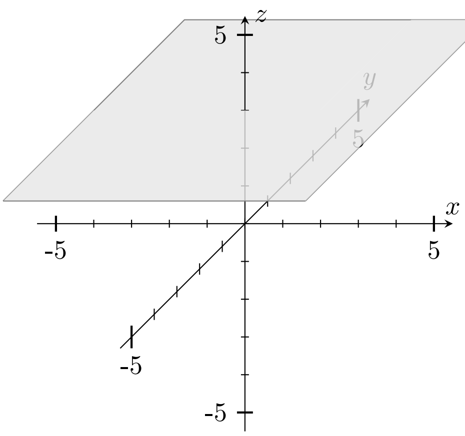
Solution: x^2+y^2 takes all possible nonegative values, so $e^{x^2+y^2}$ takes all values in $[1,\infty)$.

(3) $\star\star$ What would the graph of $z = \sqrt{1-x^2-y^2}$ look like?

Solution: This is the same as $x^2+y^2+z^2 = 1$ with $z \geq 0$, so the graph would be the upper half of the sphere of radius 1.

(4) $\star$ Which plane is this?

Solution: (C) $z = 3$.



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

2. Partial derivatives

(5) (a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

Solution: $\frac{df}{dx} = 4x$.

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

Solution: $\frac{df}{dx} = 4x$.

(c) $\star$ Let $f(x,y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

Solution: $\frac{\partial f}{\partial x} = 4x$.

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(d) $\star$ What is $\frac{\partial f}{\partial y}$?
Solution: $\frac{\partial f}{\partial y} = -2y$.

(6) Find the partial derivatives with respect to both x, y of

(a) $\star$ $g(x,y) = 3y^2 \sin(x+3)$
Solution: $\frac{\partial g}{\partial x} = 3y^2 \cos(x+3)$ (note that $3y^2$ is *constant* if y is) while $\frac{\partial g}{\partial y} = 6y\sin(x+3)$ (note that $\cos(x+3)$ is constant when x is constant).

(b) $\star$ $h(x,y) = ye^{Axy} + B$
Solution: We have $\frac{\partial h}{\partial x} \overset{\text{linear}}{=} y\left(\frac{\partial}{\partial x} e^{Axy}\right) + \frac{\partial}{\partial x} B \overset{\text{chain}}{=} y \cdot Ay \cdot e^{Axy} = Ay^2 e^{Axy}$ and $\frac{\partial h}{\partial y} \overset{\text{pdt}}{=} \left(\frac{\partial}{\partial y} y\right) \cdot e^{Axy} + y\left(\frac{\partial}{\partial y} e^{Axy}\right) = e^{Axy} + Axye^{Axy} = e^{Axy}(1 + Axy)$.

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x,y,z) = -\frac{GM}{r}$ where $r = \sqrt{x^2+y^2+z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) $\star$ The x-component of the field is given by the formula $F_x(x,y,z) = -\frac{\partial U}{\partial x}$. Find F_x
Solution: We have

$$\begin{aligned} F_x &= GM\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2} \\ &= -\frac{GM}{2}\left(x^2+y^2+z^2\right)^{-3/2} 2x \\ &= -GM\left(x^2+y^2+z^2\right)^{-3/2} \cdot x \\ &= -\frac{GM}{r^3} x\,. \end{aligned}$$

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2}$. How does it decay as a function of r?
Solution: Let $r = \left(x^2+y^2+z^2\right)^{1/2}$. Then $F_x = -\frac{GMx}{r^3}$ so $F_y = -\frac{GMy}{r^3}$ and $F_z = -\frac{GMz}{r^3}$ and we get:

$$\begin{aligned} \left|\vec{F}\right|^2 &= \frac{(GM)^2x^2}{r^6} + \frac{(GM)^2y^2}{r^6} + \frac{(GM)^2z^2}{r^6} \\ &= (GM)^2\frac{x^2+y^2+z^2}{r^6} \\ &= \frac{r^2}{r^6} = (GM)^2 r^{-4}\,. \end{aligned}$$

Thus

$$\left|\vec{F}\right| = \frac{GM}{r^2}\,.$$

This is the inverse square law.

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N,V,T) = Nk\log\left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right]\,.$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.
Solution: We have $S = Nk\log V + \frac{Nk}{\gamma-1}\log T - Nk\log N - Nk\log\Phi$

$$\begin{aligned} T\frac{\partial S}{\partial T} &= T\frac{Nk}{(\gamma-1)T} \\ &= \frac{Nk}{\gamma-1}\,. \end{aligned}$$

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping** P **constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

Solution: Substituting $V = \frac{NkT}{P}$ we get $S = Nk\log\left[\frac{kT^{\gamma/(\gamma-1)}}{\Phi P}\right] = -Nk\log P + \frac{\gamma Nk}{\gamma-1}\log T - Nk\log\frac{k}{\Phi}$ so now

$$\begin{aligned} T\frac{\partial S}{\partial T} &= T\frac{\gamma Nk}{(\gamma-1)T} \\ &= \frac{\gamma}{\gamma-1}Nk\,. \end{aligned}$$

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y\partial x}f$. Evaluate:

(a) $\star$ $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$
Solution: We have $\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}\left(Ay^2e^{Axy}\right) = A^2y^3e^{Axy}$.

(b) $\star$ $h_{xy} = \frac{\partial^2 h}{\partial y\partial x} =$
Solution: We have $\frac{\partial^2 h}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial y}\left(Ay^2e^{Axy}\right) = 2Aye^{Axy} + A^2xy^2e^{Axy} = \left(2Ay + A^2xy^2\right)e^{Axy}$.

(c) $\star$ $h_{yx} = \frac{\partial^2 h}{\partial x\partial y} =$
Solution: We have $\frac{\partial^2 h}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial x}\left(e^{Axy}\left(1+Axy\right)\right) = Aye^{Axy}\left(1+Axy\right) + e^{Axy}\cdot Ay = e^{Axy}\left(A^2xy^2 + 2Ay\right)$.

(d) $\star$ $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$
Solution: We have $\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^{Axy}\left(1+Axy\right)\right) = Axe^{xy}\left(1+Axy\right) + e^{Axy}\left(Ax\right) = Ax\left(2+Axy\right)e^{Ax}$.

(10) $\star$ Repeat this exercise for the function g from problem 2(a).
Solution: We have

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(3y^2\cos(x+3)\right) = -3y^2\sin(x+3) \\ \frac{\partial^2 g}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}\left(3y^2\cos(x+3)\right) = 6y\cos(x+3) \\ \frac{\partial^2 g}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}\left(6y\sin(x+3)\right) = 6y\cos(x+3) \\ \frac{\partial^2 g}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(6y\sin(x+3)\right) = 6\sin(x+3)\,. \end{aligned}$$

(11) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street
(say oriented toward the south), and let the y axis run across the street. Let $z = z(x,y)$ denote the height of
the street surface above sea level.

(a) $\star$ What does $\frac{\partial z}{\partial y} = 0$ say about the street?
Solution: The street surface is level.

(b) $\star$ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?
Solution: The street has a 15% grade sloping up toward the south: for each $1m$ we walk south we gain $0.15m$ in altitude.

(c) $\star$ You want to follow the street downhill. Which way should you go?
Solution: Since altitude increases with increasing x (i.e. as you go south), you should go north.

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum.
What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?
Solution: Both derivatives must be zero, else the street would be sloped in some direction.

3. Bonus (<u>**nonexaminable!**</u>): multivariable linear and higher approximation

Definition 1. A function $f(x, y)$ is *differentiable* at x_0, y_0 if we have a linear approximation $f(x, y) = f(x_0, y_0)+A(x-x_0)+B(y-y_0)+\text{small}$ as $(x, y) \to (x_0, y_0)$. We then have $A = \frac{\partial f}{\partial x}(x_0, y_0)$ and $B = \frac{\partial f}{\partial y}(x_0, y_0)$. The definition for functions of more than two variables is analogous.

(12) Let $f(x) = \sqrt{2 + x^2 + y^2}$.

(a) Write the linear approximation to f about $(1, 1)$ and use that to estimate $f(1.1, 1.2)$.
Solution: We have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{2+x^2+y^2}}$ and $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{2+x^2+y^2}}$. So at $(1, 1)$ we have $f(1, 1) = 2$, $f_x(1, 1) = f_y(1, 1) = \frac{1}{2}$ and the linear approximation is

$$f(x, y) \approx 2 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1)\,.$$

In particular

$$f(1.1, 1.2) \approx 2 + \frac{1}{2}\cdot\frac{1}{10} + \frac{1}{2}\cdot\frac{2}{10} = 2\frac{3}{20} = 2.15\,.$$

(b) Write the linear approximation to f about $(3, 5)$ and use that to estimate $f(2.8, 4.9)$.
Solution: At $(3, 5)$ we have $f(3, 5) = 6$, $f_x(3, 5) = \frac{3}{6} = \frac{1}{2}$ and $f_y(3, 5) = \frac{5}{6}$. Thus the linear approximation is

$$f(x, y) \approx 6 + \frac{1}{2}(x-3) + \frac{5}{6}(y-5)\,.$$

In particular

$$f(2.8, 4.9) \approx 6 + \frac{1}{2}\left(-\frac{2}{10}\right) + \frac{5}{6}\left(-\frac{1}{10}\right) = 6 - \frac{1}{10} - \frac{1}{12} = 5\frac{49}{60}\,.$$