Math 100A – SOLUTIONS TO WORKSHEET 2
LIMITS AND ASYMPTOTES

(1) Review of asymptotics: analyze the expression $\frac{e^x+A\sin x}{e^x-x^2}$ as $x\to\infty$, $x\to 0$, $x\to-\infty$.
Solution: This is a ratio. As $x\to\infty$ e^x grows rapidly while $A\sin x$ is bounded, so $e^x+A\sin x\sim e^x$, while in the denominator e^x dominates x^2 so $e^x-x^2\sim e^x$ and we get $\frac{e^x+A\sin x}{e^x-x^2}\sim 1$. As $x\to 0$ $e^x+A\sin x$ is close to $1+0=1$ and e^x-x^2 is close to $1-0=1$ so $\frac{e^x+A\sin x}{e^x-x^2}\sim 1$ in that regime. Finally as $x\to-\infty$ e^x decays rapidly, so $e^x-x^2\sim -x^2$ which is large. But $A\sin x$ oscillates so there is no clear asymptotic.

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.
(a) $\lim_{x\to 5}\left(x^3-x\right)$
Solution: When the function is defined by expression the limit can be obtained by plugging in. $\lim_{x\to 5}\left(x^3-x\right)=125-5=120$.
(b) $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases}\sqrt{x} & 0\le x<1\\ 3 & x=1\\ 2-x^2 & x>1\end{cases}$.
Solution: $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2-x^2)=2-1^2=1$ and $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\sqrt{x}=\sqrt{1}=1$ so

$$\lim_{x\to 1} f(x)=1\,.$$

(c) $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases}\sqrt{x} & 0\le x<1\\ 1 & x=1\\ 4-x^2 & x>1\end{cases}$.
Solution: $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4-x^2)=4-1^2=3$ and $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\sqrt{x}=\sqrt{1}=1$ so the limit does not exist (but the one-sided limits do).

(3) Let $f(x)=\frac{x-3}{x^2+x-12}$.
(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?
Solution: $f(x)=\frac{x-3}{(x-3)(x+4)}=\frac{1}{x+4}$ so $\lim_{x\to 3} f(x)=\frac{1}{3+4}=\boxed{\frac{1}{7}}$.
(b) What about $\lim_{x\to -4} f(x)$?
Solution: The limit does not exist: if x is very close to -4 then $x+4$ is very small and $\frac{1}{x+4}$ is very large. That said, when $x>-4$ we have $\frac{1}{x+4}>0$ and when $x<-4$ we have $\frac{1}{x+4}<0$ so (in the extended sense)

$$\lim_{x\to -4^+}\frac{1}{x+4}=+\infty$$
$$\lim_{x\to -4^-}\frac{1}{x+4}=-\infty\,.$$

More on this in the next lecture.

(4) Evaluate
(a) $\lim_{x\to\infty}\frac{e^x+A\sin x}{e^x-x^2}$
Solution: By problem 1 this is 1.
(b) $\lim_{x\to 0}\frac{e^x+A\sin x}{e^x-x^2}$
Solution: By problem 1 this is 1 also.

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(c) $\lim_{x\to-\infty} \frac{e^x + A\sin x}{e^x - x^2}$
Solution: By problem 1 the numerator is bounded while the denominator grows like x^2, so the whole expression tends to 0.

(5) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$
Solution: The expression is well-behaved at $x = 2$ so $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$.

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.
Solution: As $x \to -3$ the numerator is close to -1 and while the denominator goes to 0 so the whole expression blows up: we have $\frac{x+2}{x+3} \sim \frac{-1}{x+3}$. Now when $x > -3$ we have $x + 3 > 0$ so the whole expression is negative and $\lim_{x\to -3^+} \frac{x+2}{x+3} = \lim_{x\to -3^+} -\frac{1}{x+3} = -\infty$.

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$
Solution: $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{e^x}{x+2} = \frac{e^1}{1+2} = \frac{e}{3}$.

(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$
Solution: As $x \to -2$ we have $\frac{e^x(x-1)}{x^2+x-2} = \frac{e^x(x-1)}{(x-1)(x+2)} = \frac{e^x}{x+2} \sim \frac{e^{-2}}{x+2}$ and the expression blows up (we have a vertical asymptote). If $x < -2$ then $x + 2 < 0$ and thus

$$\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2} = -\infty\,.$$

(e) $\lim_{x\to 1} \frac{1}{(x-1)^2}$
Solution: The function blows up at both sides, and remains positive on both sides. Therefore

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty\,.$$

(f) $\lim_{x\to 4} \frac{\sin x}{|x-4|}$
Solution: $|x - 4| \to 0$ as $x \to 4$ while $\sin x \xrightarrow[x\to 4]{} \sin 4 \neq 0$, so the function blows up there. Since $|x - 2|$ is positive and $\sin 4$ is negative ($\pi < 4 < 2\pi$) we have

$$\lim_{x\to 4} \frac{\sin x}{|x-4|} = -\infty\,.$$

(g) $\lim_{x\to \frac{\pi}{2}^+} \tan x$, $\lim_{x\to \frac{\pi}{2}^-} \tan x$.
Solution: We have $\tan x = \frac{\sin x}{\cos x}$. Now for x close to $\frac{\pi}{2}$, $\sin x$ is close to $\sin\frac{\pi}{2} = 1$, so $\sin x$ is positive. On the other hand $\lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0$ so $\tan x$ blows up there. Since $\cos x$ is decreasing on $[0, \pi]$ it is positive if $x < \frac{\pi}{2}$ and negative if $x > \frac{\pi}{2}$, so:

$$\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$$
$$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$$

2. Limits at infinity

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3}$
Solution: As $x \to \infty$ we have $\frac{x^2+1}{x-3} \sim \frac{x^2}{x} \sim x$ so $\lim_{x\to\infty} \frac{x^2+1}{x-3} = \infty$.

(b) (Final, 2015) $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8}$
Solution: As $x \to -\infty$ we have $\frac{x+1}{x^2+2x-8} \sim \frac{x}{x^2} \sim \frac{1}{x}$ so $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8} = 0$.

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$

Solution: As $x \to -\infty$ since $\sqrt{x^2} = |x| = -x$ we have

$$\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{\sqrt{4x^2}-2x} \sim \frac{3x}{2|x|-2x}$$

$$\sim \frac{3x}{2(-x)-2x} \sim \frac{3x}{-4x} = \boxed{-\frac{3}{4}}.$$

and hence $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = -\frac{3}{4}$.

Solution: Change variables via $x = -y$ with $y \to \infty$. We are then looking at

$$\frac{-3y}{\sqrt{4y^2-y}+2y} \sim -\frac{3y}{\sqrt{4y^2}+2y} \sim -\frac{3y}{2y+2y}$$

$$\sim -\frac{3y}{4y} \sim \boxed{-\frac{3}{4}}.$$

and hence $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = -\frac{3}{4}$.