1. Asympototics (6/9/2023)

Today's Goals.

(1) Power laws, exponentials, and their asymptotics
(2) Asymptotics of sums
(3) Asymptotics of expressions

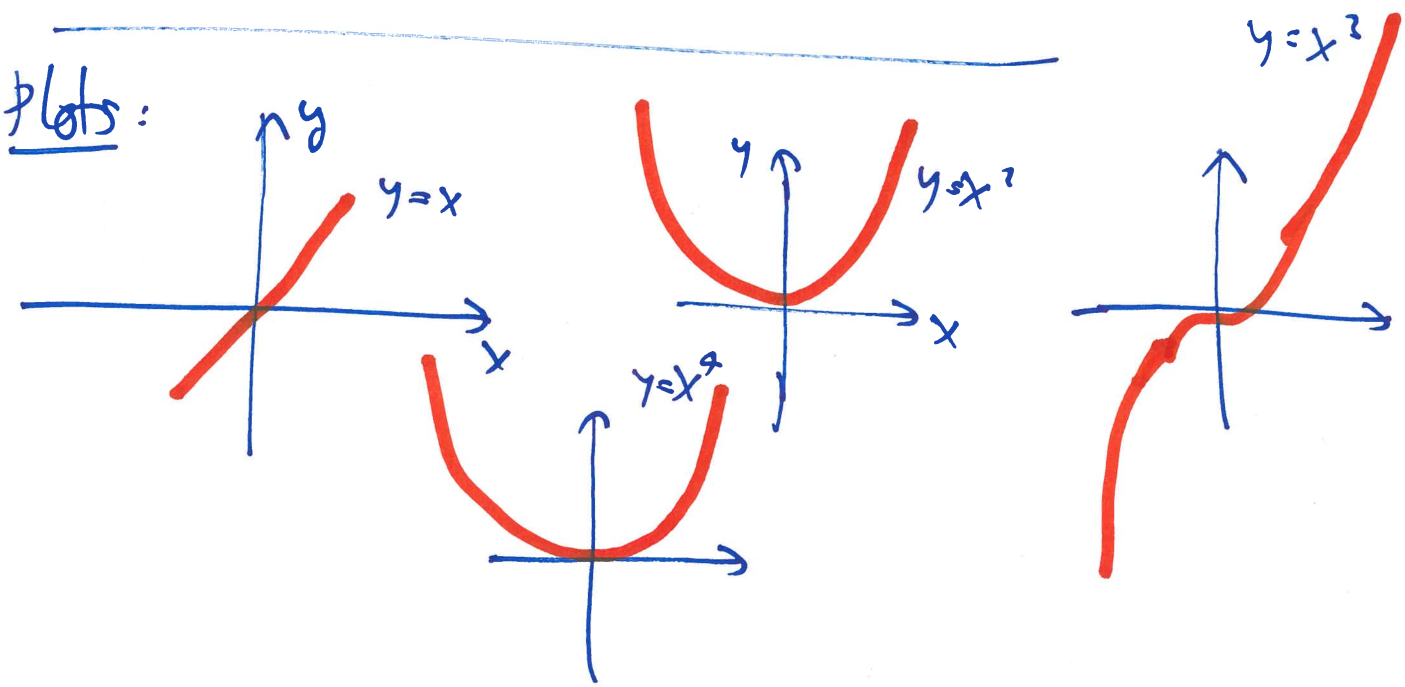
Today: look at growth & decay, Two basic patterns for this: (1) power laws , x^3, $7x^{-2}$, ... index

(2) exponentials $\frac{1}{3}\cdot 4^x$, e^x, ... base

e^{-x}, $(\frac{1}{4})^x$

WS 1

Plots:



Math 100C – WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. Asymptotics: simple expressions

(1) ⋆ Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

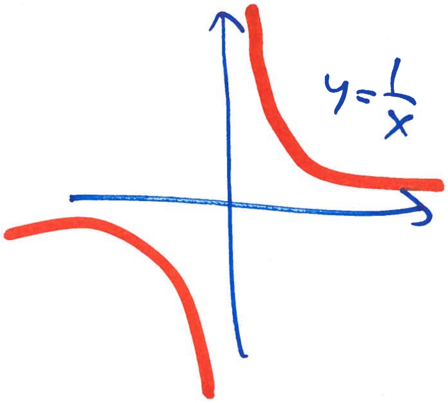
power laws: x^3, πx^{102}, $c\sqrt{x} = c \cdot x^{\frac{1}{2}}$, $-8 \cdot x^{-1}$, $9x^{-7/2}$, x^e, Ax^{-b}

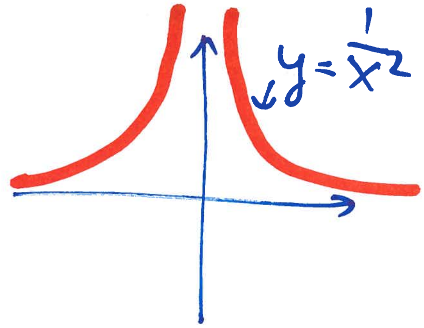
exp: $e^{2x} = (e^2)^x$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}}\left(\frac{1}{2}\right)^x$, π^x,

exp. growth rate

base

Date: 6/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.





<u>Conclusion:</u> have a "growth/decay" ladder

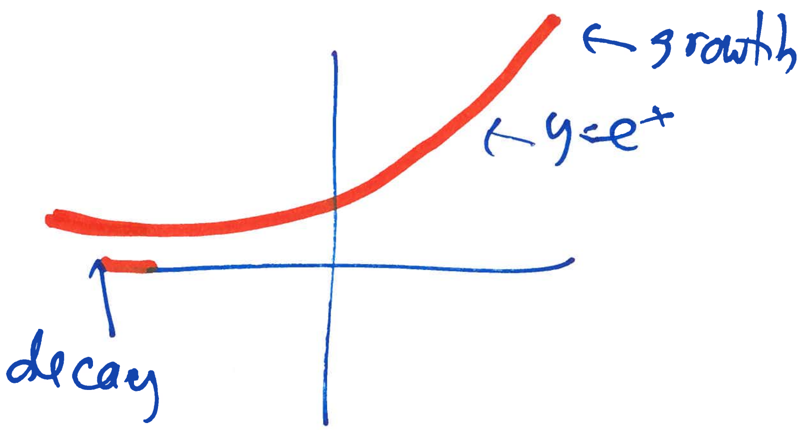
e.g. x^7 grows faster than x^2 at ∞

<u>Exponentials</u> grow faster than power laws

decay " " " "

$\frac{1}{1000}\cdot 2^x$ will (eventually) beat $10^6\cdot x^{1000}$

$10^6 e^{-x}$ " " be smaller than $\frac{1}{1000x}$



(2) Combining effects

Key idea: often when we add $f+g$, in some asymptotic regime we'll have that f dominates g (is much larger than g).

~~We~~ Example: As $x \to \infty$, $1+x^2 \sim x^2$

read "is asymptotic to"

Also as $x \to \infty$ $2x^2 + x^3 \sim x^3$

As $x \to 0$, $1+x^2 \sim 1$ on the other hand

WS 2

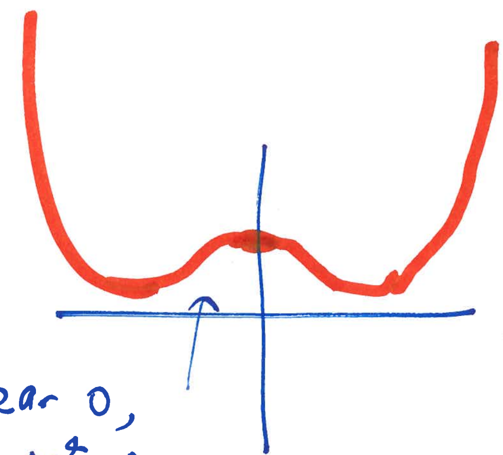
(2) $\star$ How does the each expression behave when x is large? small? what is x is large but negative? $\star\star$ Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")

As $x \to \infty$, $1 - x^2 + x^4 \sim x^4$

$x \to -\infty$ $1 - x^2 + x^4 \sim x^4$

As $x \to 0$ $1 - x^2 + x^4 \sim 1$

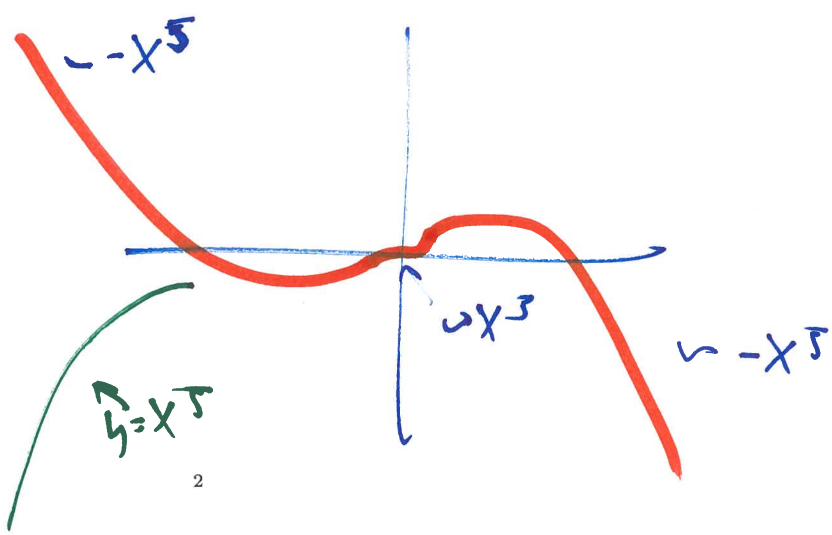


near 0, $1 - x^2 + x^4 < 1$ since $x^2 > x^4$ if x is small enough

(b) $x^3 - x^5$

As $x \to \pm\infty$, $x^3 - x^5 \sim -x^5$

$x \to 0$, $x^3 - x^5 \sim x^3$



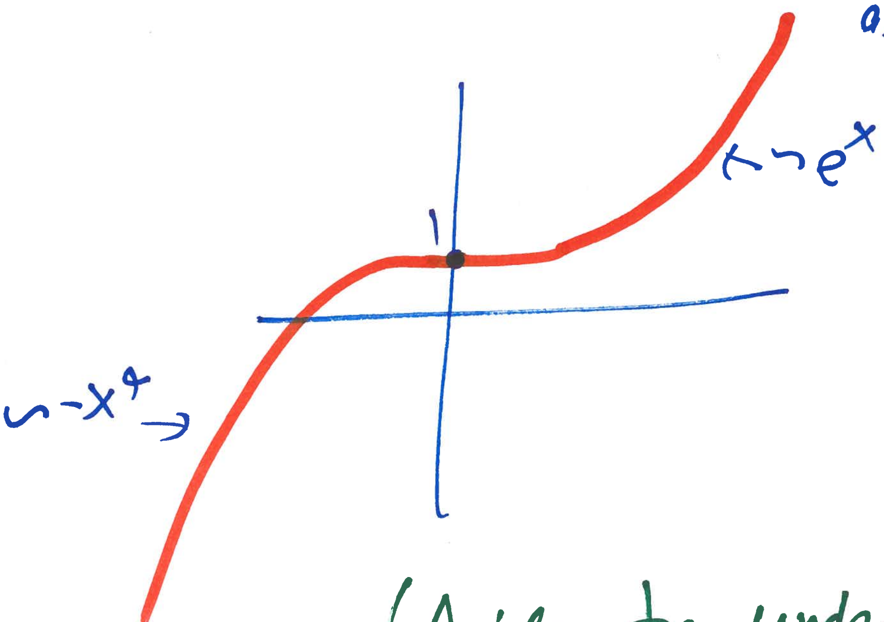
(d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A \cdot (1.02)^t$ where t is measured in years and A is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B \cdot (1.04)^t$ where B is the cost today). Suppose that today's workers can afford their healthcare (A is much bigger than B). Will that be always true? Why or why not?

(c) $e^x - x^4$

as $x \to \infty$, $e^x - x^4 \sim e^x$

as $x \to 0$ $e^x - x^4 \sim 1$ ($e^x \to 1$, x^4 small)

as $x \to -\infty$ $e^x - x^4 \sim -x^4$ (e^x decays as $x \to -\infty$)



(Aside: to understand how $e^x - x^4$ approches 1 as $x \to 0$, study $e^x - x^4 - 1$.

Facts $e^x - x^4 - 1 \sim x$ as $x \to 0$

so $e^x - x^4 \approx 1 + x$ as $x \to 0$)

(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$(*) = A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \,.$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

we'll have $(*) \sim B \cdot 1.1^t$ as $t \to \infty$

2. Asymptotics of complicated expressions

(4) Describe the following expressions in words

(a) $e^{|x-5|^3}$

Summary: (1) expressions like x^a, b^x can grow, decay

(2) If we add two expression, the asymptotically larger one dominates

(3) If we multiply/divide the sizes multiply/divide

(b) $\frac{1+x}{1+2x-x^2}$

As $x \to \infty$ $\quad 1+x \sim x$, $\quad 1+2x-x^2 \sim -x^2$ so $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} \sim$