1A)

13. Multivariable optimization (29/11/2023)

Goals.

(1) Critial points in 2d
(2) Multivariable optimization
(3) Constrained optimization

Last Time. Multivariable functions

(1) points, lines, planes in 3d
sketching 3d objects in 2d
(2) Graphs $z=f(x,y)$
(3) Partial derivatives $\frac{\partial f}{\partial x}, \ldots$
(4) Critical points: all partial derivatives vanish.

Math 100A – WORKSHEET 13
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) ⋆How many critical points does $f(x, y) = x^2 - x^4 + y^2$ have?

The derivatives are $\frac{\partial f}{\partial x} = 2x - 4x^3 = 2x(1-2x^2)$

$\frac{\partial f}{\partial y} = 2y$

the critical points all have $y=0$, $x \in \{0, \pm\frac{1}{\sqrt{2}}\}$
so have three crit. pts., at $(0,0)$, $(\pm\frac{1}{\sqrt{2}}, 0)$

(solved system $\begin{cases} 2x(1-2x^2)=0 \\ 2y=0 \end{cases}$

Date: 28/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) ⋆Find the critical points of $f(x, y) = x^2 - x^4 + xy + y^2$.

$$\frac{\partial f}{\partial x} = 2x - 4x^3 + y$$

$$\frac{\partial f}{\partial y} = x + 2y.$$

Critical pts at $\begin{cases} 2x - 4x^3 + y = 0 \\ x + 2y = 0 \end{cases}$

then $x = -2y$ so

$$-4y + 32y^3 + y = 0$$

so $-y(3 - 32y^2) = 0$ so have $y \in \{0, \pm\sqrt{3/32}\}$

So pts are $(0,0)$, $\left(-2\sqrt{3/32}, \sqrt{3/32}\right)$, $\left(2\sqrt{3/32}, -\sqrt{3/32}\right)$.

(3) (MATH 105 Final, 2013) ⋆ Find the critical points of $f(x, y) = xye^{-2x-y}$.

$$\frac{\partial f}{\partial x} = ye^{-2x-y} - 2xye^{-2x-y} = y(1-2x)e^{-2x-y}$$

$$\frac{\partial f}{\partial y} = xe^{-2x-y} - xye^{-2x-y} = x(1-y)e^{-2x-y}$$

So critical pts are over $\begin{cases} y(1-2x) = 0 \\ x(1-y) = 0 \end{cases}$

If $y = 0$, $x = 0$, if $y \neq 0$ then $1 - 2x = 0$, So $x = \frac{1}{2}$

So $\frac{1}{2}(1-y) = 0$ so $y = 1$.

$\Rightarrow$ critical pts are over $(0,0)$, $\left(\frac{1}{2}, 1\right)$.

(4)

(a) ⋆⋆ Let $f(x,y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

$\frac{\partial f}{\partial x} = 8x$ $\frac{\partial f}{\partial y} = 16y$ critical pt at $(0,0)$

local min: global min since $4x^2 + 8y^2 \geq 0$

(b) (MATH 105 Final, 2017) ⋆⋆ Let $f(x,y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

$\frac{\partial f}{\partial x} = -8x$, $\frac{\partial f}{\partial y} = 16y$ critical pt over $(0,0)$

saddle point since for fixed $y=0$ $-4x^2-3$ has local max at $x=0$, for fixed $x=0$, $8y^2-3$ has local min at $y=0$.

Multivariable optimization

"Closed interval method": ① If f defined on domain Ω has global max/min then it must occur at one of:

(a) critical pts in interior
(b) singular pts in interior
(c) on boundary

② If f is cts, Ω is closed, bounded then f has global max & min.

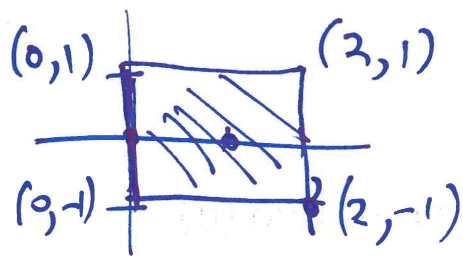
Examples: 1-vars interval $[a,b]$ has boundary $\{a,b\}$

2-var: disk $\{x^2+y^2 \le 7\}$ has boundary circle $\{x^2+y^2=7\}$

boundary of domains in plane are often curves

For $f(x,y)$ optimization on boundary curves is 1-var optimization.

(5) ⋆ Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.



2. Optimization

(6) ⋆⋆Find the minimum of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$:

(a) on the rectangle $0 \le x \le 2$, $-1 \le y \le 1$. **min = −7**

$\frac{\partial f}{\partial x} = 4x - 4 = 4(x-1)$, $\frac{\partial f}{\partial y} = 6y$ so critical pt $(1, 0, -7)$

On $x=0$, $f(0,y) = 3y^2 - 5$ has min -5 at $y=0$ $\}$ $-1 \le y \le 1$

On $x=2$ $f(2,y) = 3y^2 - 5$ " " " " "

On $y = \pm 1$, $f(x, \pm 1) = 2x^2 - 4x - 2 = 2(x^2 - 2x + 1) - 4 = 2(x-1)^2 - 4$ has min -4 at $x=1$ (on $0 \le x \le 2$)

(b) on the rectangle $2 \le x \le 3$, $-1 \le y \le 1$.

No critical pts: $(1,0) \notin$ rectangle

on $x=2$, $f(2,y) = 3y^2 - 5$, min at $y=0$ of -5

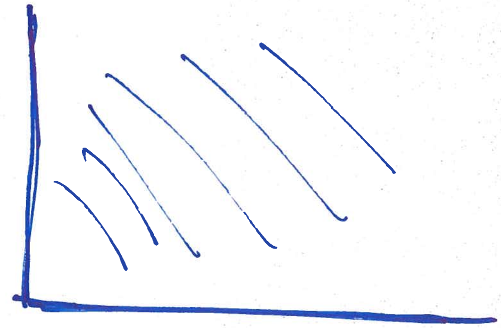
on $x=3$, $f(3,y) = 3y^2 + 1$, " " " " 1.

On $y = \pm 1$, $f(x, \pm 1) = 2(x-1)^2 - 4$ for $2 \le x \le 3$ min of -2 at $x=2$

over all min $= -5$, attained at $(2, 0)$.

(7) Find the maximum of $(7x+3y+2y^2)e^{-x-y}$ for $x \geq 0$, $y \geq 0$,

domain: first quadrant



toward ∞:

as $x \to \infty$, e^{-x} decays, dominates $7x$, as $y \to \infty$ e^{-y} decays, dominates $3y, 2y^2$.

So as $x \to \infty$ or $y \to \infty$, function tends to 0

But for any $x, y > 0$ $f(x,y) > 0$, so have max, either at critical pts or on $x=0$, or on $y=0$

$$\frac{\partial f}{\partial x} = 7e^{-x-y} - (7x+3y+2y^2)e^{-x-y} = (7-7x-3y-2y^2)e^{-x-y}$$

$$\frac{\partial f}{\partial y} = (3+4y)e^{-x-y} - (7x+3y+2y^2)e^{-x-y} = (3+4y-7x-3y-2y^2)e^{-x-y}$$

crit. pts over solution to $\begin{cases} 7 = 7x+3y+2y^2 \\ 3+4y = 7x+3y+2y^2 \end{cases}$ ($e^{-x-y} \neq 0$)

The system implies $7 = 3+4y$ so $y=1$, $7 = 7x+3+2$

so $x = \frac{2}{7}$, ie. at $(\frac{2}{7}, 1, 7e^{-1\frac{2}{7}})$

boundary: on $x=0$, function is: $(3y+2y^2)e^{-y}$:
At $y=0$ get 0, as $y\to\infty$ limit is 0, max in betwe

$$\frac{d}{dy}(3y+2y^2)e^{-y} = (3+4y-3y-2y^2)e^{-y}$$
$$= (3+y-2y^2)e^{-y}$$

Crit pts if $2y^2-y-3=0$, $y = \frac{1\pm\sqrt{1+24}}{4} = \frac{1\pm5}{4} = \frac{3}{2},$

at $x=0$, $y=\frac{3}{2}$ have value $\left(\frac{9}{2}+\frac{9}{2}\right)e^{-3/2} = 9e^{-3/2}$

On $y=0$, function is $7xe^{-x}$ again at $x=0$ as $x\to\infty$ get 0,

$\frac{d}{dx}(7xe^{-x}) = (7-7x)e^{-x}$ vanishes at $x=1$

where value is $7\cdot e^{-1}$

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max value is one of $7e^{-9/7}$, $9e^{-3/2}$, $7e^{-1}$

$7e^{-9/7} < 7e^{-1}$, so $7e^{-9/7}$ eliminated to compare

$9e^{-3/2}$, $7e^{-1}$ compare $\frac{81}{e^3}$ to $\frac{49}{e^2}$, compare $\frac{81}{49}$ to e

note $\frac{81}{49} < 2$, while $e > 2$

so $7e^{-1} > 9e^{-3/2}$, so max is $\frac{7}{e}$ at $(1,0)$

(8) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C = 3t^2 + \sqrt{t} \cdot q$. At price p the company can sell $q = \frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(~~x~~ p, t)$.

at p, t make $\frac{t-p}{3}$ widgets, get $p\frac{(t-p)}{3}$ revenue

profit $p\frac{(t-p)}{3} - 3t^2 - \sqrt{t} \cdot \frac{t-p}{3}$

(b) How many widgets of what quality should the company make to maximize profits?

need to maximize $f(p,t) = \frac{1}{3}p(t-p) - 3t^2 - \sqrt{t}\,\frac{t-p}{3}$

on $t \geq 0$, $0 \leq p \leq t$