P1. Recall that a projection is a linear map $E$ such that $E^2 = E$. For each $n$ construct a projection $E_n : \mathbb{R}^2 \to \mathbb{R}^2$ of norm at least $n$ ($\mathbb{R}^n$ is equipped with the Euclidean norm unless specified otherwise). Prove for yourself that the norm of an orthogonal projection is 1.

**Difference and Differential Equations**

P2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let $v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find $S$ invertible and $D$ diagonal such that $A = S^{-1}DS$.

(b) Find a formula for $v_k = A^k v_0$, and show that $\frac{v_k}{\|v_k\|}$ converges for any norm on $\mathbb{R}^2$.

RMK You have found a formula for Fibonacci numbers (why?), and have shown that the real number $\frac{1}{2} \left(1 + \sqrt{5} \right)^n$ is exponentially close to being an integer.

RMK This idea can solve any difference equation (see PS10). We now apply this to solving differential equations.

1. We will analyze the differential equation $u'' = -u$ with initial data $u(0) = u_0$, $u'(0) = u_1$.

(a) Let $\psi(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$. Show that $u$ is a solution to the equation iff $\psi$ solves

$$\psi'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi(t).$$

(b) Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find formulas for $W^n$ and express $\exp(Wt) = \sum_{k=0}^{\infty} \frac{W^k t^k}{k!}$ as a matrix whose entries are standard power series.

(c) Sum the series and show that $u(t) = u_0 \cos(t) + u_1 \sin(t)$.

(d) Find a matrix $S$ such that $W = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} S^{-1}$. Evaluate $\exp(Wt)$ again, this time using $\exp(Wt) = S \left( \exp \left( \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \right) \right) S^{-1}$.

2. Consider the differential equation $\frac{d}{dt} \psi = By$ where $B$ is at in PS7 problem 1.

(a) Find matrices $S,D$ so that $D$ is in Jordan form, and such that $B = SDS^{-1}$.

(b) Find $\exp(tD)$ as in 1(b) by computing a formula for $D^n$ and summing the series.

(c) Find the solution such that $\psi(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^t$.

3. Let $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with $z \in \mathbb{C}$.

(a) Find (and prove) a simple formula for the entries of $A^n$.

(b) Use your formula to decide the set of $z$ for which $\sum_{n=0}^{\infty} A^n$ converge, and give a formula for the sum.

(c) Show that the sum is $(\text{Id} - A)^{-1}$ when the series converges.
Extra credit

4. For any matrix $A$ show that $\sum_{n=0}^{\infty} z^n A^n$ converges for $|z| < \frac{1}{\rho(A)}$.

Supplementary problems

A. Consider the map $\text{Tr}: M_n(F) \to F$.
   (a) Show that this is a continuous map.
   (b) Find the norm of this map when $M_n(F)$ is equipped with the $L^1 \to L^1$ operator norm (see PS8 Problem 2(a)).
   (c) Find the norm of this map when $M_n(F)$ is equipped with the Hilbert–Schmidt norm (see PS8 Problem 4).
   (*) (d) Find the norm of this map when $M_n(F)$ is equipped with the $L^p \to L^p$ operator norm.

Find the matrices $A$ with operator norm 1 and trace maximal in absolute value.

B. Call $T \in \text{End}_F(V)$ bounded below if there is $K > 0$ such that $\|Tv\| \geq K\|v\|$ for all $v \in V$.
   (a) Let $T$ be bounded below. Show that $T$ is invertible, and that $T^{-1}$ is a bounded operator.
   (*) (b) Suppose that $V$ is finite-dimensional. Show that every invertible map is bounded below.

C. (The supremum norm and the Weierstrass $M$-test) Let $V$ be a complete normed space.

DEF For a set $X$ call $f: X \to V$ bounded if there is $M > 0$ such that $\|f(x)\|_V \leq M$ for all $x \in X$ in which case we write $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_V$ (equivalently, $f$ is bounded if $x \mapsto \|f(x)\|_V$ is in $\ell^\infty(X)$).
   (a) Show that $\ell^\infty(X; V)$ is a vector space (this doesn’t use completeness of $V$).
   (b) Show that $\ell^\infty(X; V)$ is complete.

DEF Now suppose that $X$ is a topological space (if you aren’t sure about this, simply assume $X \subset \mathbb{R}^n$). Let $C(X; V)$ denote the space of continuous functions $X \to V$ and let $C_b(X; V) = C(X; V) \cap \ell^\infty(X; V)$ be the space of bounded continuous functions, the latter equipped with the $\ell^\infty$-norm.
   (c) Show that $C_b(X; V)$ is complete (equivalently, that it is a closed subspace of $\ell^\infty(X; V)$).

COR Deduce Weirestrass’s $M$-test: $f_n: X \to V$ are continuous and $\|f_n\|_\infty \leq M_n$ with $\sum_n M_n < \infty$ then $\sum_n f_n$ converges to a continuous function bounded by $\sum_n M_n$. 

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