Lior Silberman’s Math 412: Problem Set 5 (due 12/10/2017)

Practice

P1. Let \( U = \text{Span}_F \{ \mathbf{u}_1, \mathbf{u}_2 \} \) be two-dimensional. Show that the element \( \mathbf{u}_1 \otimes \mathbf{u}_1 + \mathbf{u}_2 \otimes \mathbf{u}_2 \in U \otimes U \) is not a pure tensor, that is not of the form \( \mathbf{u} \otimes \mathbf{u} \) for any \( \mathbf{u} \in U \).

P2. Let \( \iota : U \times V \rightarrow U \otimes V \) be the standard inclusion map \( (\iota(\mathbf{u}, \mathbf{v}) = \mathbf{u} \otimes \mathbf{v}) \). Show that \( \iota(\mathbf{u}, \mathbf{v}) = 0 \) if \( \mathbf{u} = 0_U \) or \( \mathbf{v} = 0_V \) and that for non-zero vectors we have \( \iota(\mathbf{u}, \mathbf{v}) = \iota(\mathbf{u}', \mathbf{v}') \) iff \( \mathbf{u}' = \alpha \mathbf{u} \) and \( \mathbf{v}' = \alpha^{-1} \mathbf{v} \) for some \( \alpha \in F^\times \).

P3. Let \( U, V \) be finite-dimensional spaces and let \( A \in \text{End}(U), B \in \text{End}(V) \).
   - (a) Construct a map \( A \oplus B \in \text{End}_F(U \oplus V) \) restricting to \( A, B \) on the images of \( U, V \) in \( U \oplus V \).
   - (b) Show that \( \text{Tr}(A \oplus B) = \text{Tr}(A) + \text{Tr}(B) \).
   - (c) Evaluate \( \det(A \oplus B) \).

Tensor products of maps

1. Let \( U, V \) be finite-dimensional spaces, and let \( A \in \text{End}(U), B \in \text{End}(V) \).
   - (a) Show that \( \mathbf{u}, \mathbf{v} \mapsto (A\mathbf{u}) \otimes (B\mathbf{v}) \) is bilinear, and obtain a linear map \( A \otimes B \in \text{End}(U \otimes V) \).
   - (b) Suppose \( A, B \) are diagonalizable. Using an appropriate basis for \( U \otimes V \), obtain a formula for \( \det(A \otimes B) \) in terms of \( \det(A) \) and \( \det(B) \).
   - (c) Extending (a) by induction, show for any \( A \in \text{End}_F(V) \), the map \( A^{\otimes k} \) induces maps \( \text{Sym}^k A \in \text{End}(\text{Sym}^k V) \) and \( \wedge^k A \in \text{End}(\wedge^k V) \).
   - (d) Show that the formula of (b) holds for all \( A, B \).

SUPP (Notation extended from supplement to PS4) Let \( V_K = K \otimes_F V \) be an extension of scalars. For \( T \in \text{End}_F(V) \) let \( T_K = \text{Id}_K \otimes T_K \). Show that \( T_K \in \text{End}_K(V_K) \), and that the natural inclusions \( \text{Ker}(T), \text{Im}(T) \subset V \) extend to identifications \( (\text{Ker}(T))_K = \text{Ker}(T_K) \) and \( (\text{Im}(T))_K = \text{Im}(T_K) \).

2. Suppose \( \frac{1}{2} \in F \), and let \( U \) be finite-dimensional. Construct isomorphisms

\[
\{ \text{symmetric bilinear forms on } U \} \leftrightarrow (\text{Sym}^2 U)' \leftrightarrow \text{Sym}^2 (U') .
\]

Nilpotence

3. Let \( U \in M_n(F) \) be strictly upper-triangular, that is upper triangular with zeroes along the diagonal. Show that \( U^n = 0 \) and construct such \( U \) with \( U^{n-1} \neq 0 \).

4. Let \( V \) be a finite-dimensional vector space, \( T \in \text{End}(V) \).
   - (a) Show that the following statements are equivalent:
     \[
     \begin{align*}
     \forall \mathbf{v} & : \exists k \geq 0 : T^k \mathbf{v} = 0 ; \quad \exists k \geq 0 : \forall \mathbf{v} \in V : T^k \mathbf{v} = 0 .
     \end{align*}
     \]
   - (b) Find nilpotent \( A, B \in M_2(F) \) such that \( A + B \) isn’t nilpotent.
   - (c) Suppose that \( A, B \in \text{End}(V) \) are nilpotent and that \( A, B \) commute. Show that \( A + B \) is nilpotent.
Extra credit

5. Let $V$ be finite-dimensional.
   (a) Construct an isomorphism $U \otimes V' \rightarrow \text{Hom}_F(V, U)$.
   (b) Define a map $\text{Tr}: U \otimes U' \rightarrow F$ extending the evaluation pairing $U \times U' \rightarrow F$.
   DEF The trace of $T \in \text{Hom}_F(U, U)$ is given by identifying $T$ with an element of $U \otimes U'$ via (a) and then applying the map of (b).
   (c) Let $T \in \text{End}_F(U)$, and let $A$ be the matrix of $T$ with respect to the basis $\{u_i\}_{i=1}^n \subset U$. Show that $\text{Tr} T = \sum_{i=1}^n A_{ii}$.
   RMK This shows that similar matrices have the same trace!
   (d) Solve P3(b) from this point of view.

Supplementary problems

A. (The tensor algebra) Fix a vector space $U$.
   (a) Extend the bilinear map $\otimes: U^\otimes n \times U^\otimes m \rightarrow U^\otimes (n+m)$ to a bilinear map $\otimes: \bigoplus_{n=0}^\infty U^\otimes n \times \bigoplus_{n=0}^\infty U^\otimes n \rightarrow \bigoplus_{n=0}^\infty U^\otimes n$.
   (b) Show that this map $\otimes$ is associative and distributive over addition. Show that $1_F \in F \simeq U^\otimes 0$ is an identity for this multiplication.
   DEF This algebra is called the tensor algebra $T(U)$.
   (c) Show that the tensor algebra is free: for any $F$-algebra $A$ and any $F$-linear map $f: U \rightarrow A$ there is a unique $F$-algebra homomorphism $\bar{f}: T(U) \rightarrow A$ whose restriction to $U^\otimes 1$ is $f$.

B. (The symmetric algebra) Fix a vector space $U$.
   (a) Endow $\bigoplus_{n=0}^\infty \text{Sym}^n U$ with a product structure as in 3(a).
   (b) Show that this creates a commutative algebra $\text{Sym}(U)$.
   (c) Fixing a basis $\{u_i\}_{i \in I} \subset U$, construct an isomorphism $F\left[\{x_i\}_{i \in I}\right] \rightarrow \text{Sym}^* U$.
   RMK In particular, $\text{Sym}^* (U')$ gives a coordinate-free notion of “polynomial function on $U$”.
   (d) Let $I \triangleleft T(U)$ be the two-sided ideal generated by all elements of the form $u \otimes v - v \otimes u \in U^\otimes 2$. Show that the map $\text{Sym}(U) \rightarrow T(U)/I$ is an isomorphism.
   RMK When the field $F$ has finite characteristic, the correct definition of the symmetric algebra (the definition which gives the universal property) is $\text{Sym}(U) \overset{\text{def}}{=} T(U)/I$, not the space of symmetric tensors.

C. Let $V$ be a (possibly infinite-dimensional) vector space, $A \in \text{End}(V)$.
   (a) Show that the following are equivalent for $v \in V$:
      (1) $\text{dim}_F \text{Span}_F \{A^n v\}_{n=0}^\infty < \infty$;
      (2) there is a finite-dimensional subspace $W \subset V$ such that $AW \subset W$.
   DEF Call such $v$ locally finite, and let $V_{\text{fin}}$ be the set of locally finite vectors.
   (b) Show that $V_{\text{fin}}$ is a subspace of $V$.
   (c) Call $A$ locally nilpotent if for every $v \in V$ there is $n \geq 0$ such that $A^n v = 0$ (condition (1) of 5(a)). Find a vector space $V$ and a locally nilpotent map $A \in \text{End}(V)$ which is not nilpotent.
   (d) $A$ is called locally finite if $V_{\text{fin}} = V$, that is if every vector is contained in a finite-dimensional $A$-invariant subspace. Find a space $V$ and locally finite linear maps $A, B \in \text{End}(V)$ such that $A + B$ is not locally finite.