Math 322, lecture 21, 21/11/2017

Today: 1) PS9 2) Solve Nilpotence

PS9, 2(a)

Let $G$ have order $255 = 3 \cdot 5 \cdot 17$.

Sylow theory: $P_3 = C_3$, $P_5 = C_5$, $P_{17} = C_{17}$,

$n_3(G) \in \{1, 5 \cdot 17\}$, $n_5(G) = \{1, 3 \cdot 17\}$, $n_{17}(G) = \{1\}$

divide $5 \cdot 17$

$b = 1 (3)$

(a) $n_{17}(G) = 1$ so $P_{17}$ is normal. Then, $g \cdot x = gxg^{-1}$ defines an action of $G$ on $P_{17}$ by automorphisms, i.e. a hom $G \to \text{Aut}(P_{17}) = \text{Aut}(C_{17}) = (\mathbb{Z}/17\mathbb{Z})^\times \cong C_{16}$

Now order of image of this hom divides $|G|

(first isom thm)$ and divides 16 (Lagrange's thm in $\text{Aut}(P_{17})$).

so the image is trivial, i.e. the action is trivial: $g \cdot x = x$ for all $g \in G$, $x \in P_{17}$, i.e. $P_{17} \subseteq Z(G)$

(b) now no "way" to have orbit of size containing 17 in the action of $G$ on itself by conjugation

Now $\text{Syl}_{17}(G)$ is a single conjugacy class, so by orbit-stabilizer thm, $n_6(G) = [G : N_G(P_{17})]$
Now $P_{17}$ is central, so acts trivially on $\text{Syl}_5(G)$, hence $P_{17} \leq N_G(P_5)$ and $\#N_G(P_5)$ is divisible by $5$ (H. Pt).

So $[G : N_G(P_5)]$ divides $3$.

But $n_5(G)$ not $3$, so $n_5(G) = 1$.

Let $G$ have order $140 = 2^2 \cdot 5 \cdot 7$.

Then $\#P_2 = 4$, $\#P_5 = 5$, $\#P_7 = 7$.

$n_2(G) \in \{1, 5, 7, 35\}$, $n_5(G) \in \{1, 7\}$, $n_7(G) \in \{1\}$.

Let $G$ have order $140$. Let $P_5, P_7$ be normal, disjoint (relatively prime orders).

$\Rightarrow$ $H = P_5P_7$ is a subgroup.

Since $P_5, P_7$ are both normal, disjoint, they commute (HW: if $x \in P_5, y \in P_7$ then $[x, y] = x y x^{-1} y^{-1} \in P_5 \cap P_7 = \{e\}$).

$\Rightarrow$ $H$ is a gp of order $35 = 5 \cdot 7$. Since $7 \nmid 1(5)$, $H = C_{35}$ (thm on pq-groups).
$H$ is normal (generated by normal subg) disjoint from $P_2$ (order of $H$ is odd).

So $P_2 \times H$ is a semidirect prod of order $4 \cdot 35 = 140$.

I.e. $G = P_2 \times H = P_2 \times C_{35}$

Remains: (1) Study Hom ($P_2$, Aut($C_{35}$)) up to automorphism

(2) Check for non-isom

For this: $\text{Aut}(C_{35}) = (\mathbb{Q}/35\mathbb{Z})^\times \times \mathbb{Z}/35\mathbb{Z}$

Respects mult.

$\cong (\mathbb{Q}/35\mathbb{Z}) \times (\mathbb{Q}/35\mathbb{Z})^\times \cong C_4 \times C_6$

$\cong C_4 \times C_2 \times C_3$

**P58, Problem 6:** up to isom, $C_4 \times C_{35}$ determined

by subgp of $C_4 \times C_6$ of order 14

subgp of of order 4, generated by $(1, 7), (6, 2), (17, [3])$

of $C_4 \times C_6$ one of

Get two subgps: $C_4 \times 35Z_7$, $(10, 5, 7), (11, 7, [3]), (7, 7, 6)$

These groups are distinct: in first case, $C_4$ only acts on $P_2$,

$C_4 \times 35, C_{35}, C_4 \times C_6, C_{35}$

In the second case no

(first case is $(C_4 \times C_{35}) \times C_7$)
Subgrp of order 2: same as elements of order 2, get:
\[[23]_4, [03]_6, ([3]_4, [3]_6), ([27]_6, [3]_6)\]
(mean: generator of \( P_2 < C_4 \) will act by one of those)

They are distinct: in first two cases \( P_2 \) will commute with \( P_3 \) or \( P_5 \), in second with neither.

Subgrp of order 4: can have \( C_4 \times C_4 = C_{16} \)

Points let \( P \leq G \leq G_0, G' \leq G_0 \) be of order 16.

Say \( P \sim P' \) (2-Sylow subgroups) occur both.

Let \( f: G \to G' \) be an isom.

Say \( G, G' \) have unique subgps of order 35, \( H, H' \),
so \( f(H) = H' \). Let \( P_2 < G \) be a 2-Sylow subgp,
then \( f(P_2) \sim P_2' \) is a 2-Sylow subgp (also has order 4).

Let \( \alpha: P_2 \to \text{Aut}(H) \) be the conjugation action
\( \alpha': P_2' \to \text{Aut}(H') \)

The \( \alpha' = f_\alpha \cdot \alpha \circ f^{-1} \), where if \( \beta \in \text{Aut}(H') \)
\( f_\alpha \beta = f \circ \beta \circ f^{-1} \)
in particular, conjugation by \( f_{H_1} \) (as a map \( \text{Aut}(H) \to \text{Aut}(H') \))
gives isom \( \alpha(P_2) \cong \alpha'(P_2') \).
Nilpotence

Motivation: Problem: Given $f \in \mathbb{Q}[x]$, find roots of $f$.

(e.g. $f(x) = x^2 + bx + c$, roots are $\frac{-b \pm \sqrt{b^2-4c}}{2}$)

(similar if $\deg f = 3$, $\alpha$'s)

Assume $f$ irreducible (no factors in $\mathbb{Q}[x]$ other than $1, f$).

Galois:

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$, $d = \deg f$.

be the roots.

Let $F = \text{smallest field of complex numbers containing}$

$\alpha_1, \ldots, \alpha_d$.

$= \text{Span}_\mathbb{Q} \{ 1, \alpha_1, \ldots, \alpha_1^j, \ldots, \alpha_d, \alpha_d^j, \ldots \}$

Facts: $\dim_{\mathbb{Q}} \ F = d$!

(example: $f(x) = x^2 + 1$, $F = \mathbb{Q}(i) = \{ a + bi \mid a, b \in \mathbb{Q} \}$

$f(x) = x^2 - 2$, $F = \mathbb{Q}(\sqrt{2}, w) = \mathbb{Q}(\sqrt{2}, \omega, \omega^{-1})$

$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\omega^3 = 1$)

Let $G = \text{Gal}(f) = \{ \sigma \in S_d \mid \text{permuting } \alpha_1, \ldots, \alpha_d \text{ respects} \}$

multiplication + addition

(e.g. if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ = 0

then must have $\alpha_{\sigma(1)}, \alpha_{\sigma(2)} + \alpha_{\sigma(3)}, \alpha_{\sigma(4)} = 0$)

(1838

Thm: (Abel) Suppose $\text{Gal}(f)$ is commutative. Then can

compute roots of $f$ using $+ - \sqrt{-}$.)
(hence commutative groups called "Abelian")

Thm (Galois, -1830): Explicit group-theoretic condition ("solvability") s.t. roots of $f$ expressible by radicals if $\text{Gal}(f)$ is solvable.

Also, $S_3, S_4$ solvable $\Rightarrow$ cubic, quartic formulas quadratic.

So if $n \geq 5$ not solvable (because $A_n$ simple if $n \geq 5$) so no quintic formula.