Thm. (Sylow) Let G be a finite group of order n, p prime s.t. $p^k | n$. Then,

1. Every $p$-subgroup of G is contained in a $p$-subgroup of order $p^k$ ("Sylow $p$-subgroup")
2. All Sylow $p$-subgroups are conjugate
3. $n_p(G) = \# Syl_p(G)$ satisfies $n_p(G) | n$, $n_p(G) \equiv 1 \pmod{p}$

Remark: If $n = p^m$, with $p \nmid m$ then, (3) says:

$$n_p(G) | m, \quad n_p(G) \equiv 1 \pmod{p} \quad (\star)$$

Basic exercise: C""numeroslogy") given $n$, list all prime $p$ dividing $n$, for each $p$ all solutions to (\star)

Example: the groups of order 12 are:

$C_{12}, C_3 \times C_2 \times C_2 = C_6 \times C_2, A_4, C_2 \times S_3, C_4 \times C_3$

(Q: which of these is $D_{12}$?)

Pf: $3 | 12$. $n_3(G)$ is a divisor of 4, and be $\equiv 1 \pmod{3}$ so $n_3(G) \in \{1, 4\}$.

Also $n_2(G) | 3$, $n_2(G) \equiv 1 \pmod{2}$ so $n_2(G) \in \{1, 3\}$

implicit: (1) there are pairwise distinct (2) there is a unique (up to isom) non-commutative pt $C_4 \times C_3$. 

C_4 \times C_3$. 

Case 2b: The action is non-trivial and $P_2 = V \cong C_2 \times C_2$

The action is a hom $\varphi: P_2 \to \text{Aut}(P_3) = \text{Aut}(C_3) = C_2$

$\varphi$ is non-trivial, so surjective ($C_2$ has only 2 subgroups)

Let $K = \ker(\varphi)$. Then $V/K = C_2$ (1st isom thm)

So $|K| = 2$. Then $K = C_2$ and taking an $e \in V \setminus K$, the group $L = G/K$ is complementary: $V = K \times L$. (And $\varphi(e)$ is non-trivial element of $\text{Aut}(C_3)$)

So $G = V \times C_3 = K \times (L \times C_3) = C_2 \times D_6 \cong C_2 \times S_3$.

Case 2c: The action is non-trivial and $P_2 = C_4$.

Say $a \in C_4$ is a generator, $\varphi: P_2 \to \text{Aut}(C_3) = C_3$ the action $C_3 \cong \mathbb{Z}/4\mathbb{Z} \cong C_4$

Then $\varphi$ is determined by $\varphi([1])$, and if $\varphi$ is non-trivial, must have $\varphi([1])$ a non-trivial element of $C_2$.

\textbf{Injective:} Because 2 (order of this element) divides 4 (order of generator of $C_4$), such $\varphi$ exists (see PS8).

So get a unique non-commuting semidirect product $C_4 \rtimes C_3$.

Concretely, this means: let $a \in C_4$ be the generator, then if $h \in C_3$, $a^i ha^{-i} = \begin{cases} h^{-1} & \text{if } i \text{ odd} \\ h & \text{if } i \text{ even} \end{cases}$
Case 1: \( n_3(G) = 4 \). We know \( G \) acts by conjugation on \( \text{Syl}_3(G) \). This is an action on a set of size 4, hence a hom \( f: G \to S_4 \).

Let \( P_3 \in \text{Syl}_3(G) \). The \( [G:N_6(P_3)] = \# \text{ conjugates of } P_3 \)

\[
\begin{align*}
4 &
\rightarrow \mathbb{N}_6(P_3) \rightarrow 12 \\
3 &
\rightarrow P_3 \rightarrow ? \rightarrow ?
\end{align*}
\]

\[ \#P_3 = 3 \]

But \( [G:P_3] = \frac{\#G}{\#P_3} = \frac{12}{3} = 4 \)

So \( [N_6(P_3):P_3] = \frac{4}{4} = 1 \)

So \( P_3 \) is its own normalizer.

\( \text{Ker}(f) \lhd P_3 \) (the kernel of action stabilizes every point)

\( P_3 \) is not normal (it has 4 conjugates)

So \( \text{Ker}(f) \neq P_3 \), so \( (P_3 \cong C_3 \text{ has no non-triv subgps}) \)

So \( \text{Ker}(f) = \{ e \} \)

Conclusion: \( f \) is an isom onto its image: \( G = \text{subgp of } S_4 \) of order 12.

Now the different 3-Sylow subgps are orthogonal:

If \( P_3, P_3' \in \text{Syl}_3(G) \), \( P_3 \cap P_3' \lhd P_3 \) if this was \( P_3 \), we'd get 8 instead \( P_3 \cap P_3' = \{ e \} \) (\( G \) only has \( P_3 \times P_3' \), so \( P_3 \neq P_3' \)

Each \( P_3 \cong C_3 \) has two non-identity elements, so \( G \) has \( 8 = 4 \cdot 2 \) elements of order 2.
So \( f(G) \leq S_4 \) has 8 elements of order 3.

Only elements of \( S_4 \) of order 3 are 3-cycles, and there are 8 of them (4 ways to choose support, 2 cyclic orderings of \((1,2,3)\)).

So the image of \( f \) contains all 3-cycles, hence the subgroup they generate: \( f(G) \leq A_4 \).

But \( |f(G)| = |G| = 12 = |A_4| \) so \( f: G \rightarrow A_4 \) is an isomorphism.

**Conclusion:** If \( n_3(G) = 4 \) then \( n_2(G) = n_2(A_4) = 3 \)

\[ P_2 \triangleright G \times G \]

**Case 2:** \( n_3(G) = 1 \). Now \( P_3 \triangleleft G \).

Let \( P_2 \in \text{Syl}_2(G) \). Then \( |P_2| = 2^2 = 4 \), so \( P_2 \cap P_3 = \{e\} \) (\( 2^2 \) and 3 have no common divisors), and \( |P_2 P_3| = |P_2| \cdot |P_3| = 4 \cdot 3 = 12 \)

So \( G = P_2 P_3 \) where \( P_2 \cap P_3 = \{e\} \), \( P_3 \triangleleft G \),

i.e. \( G \) is a semidirect product of a group of order 4 and a group of order 3.

It remains to classify actions of \( P_2 \) on \( P_3 \).

**Case 2a:** the action is trivial; \( P_2, P_3 \) commute:

\[ G = P_2 \times P_3 \] i.e. \( G = C_4 \times C_3 = C_{12} \)

or

\[ G = (C_2 \times C_2) \times C_3 = C_2 \times C_6 \]

and they are distinct since they have non-isomorphic \( P_2 \)'s.
Fact: \( \text{Aut}(C_p) = (\mathbb{Z}/p\mathbb{Z})^* = \mathbb{Z}/(p-1)\mathbb{Z} \cong C_{p-1} \). If \( p \) is prime (if \( p \) is odd, \( \text{Aut}(C_{p^k}) = C_{p-1} \cdot p^{k-1} \))

Example: There is no simple group of order 30.

Pf: Suppose \( G \) was a simple group of order 30.

Numerology gives \( n_3(G) \in \{1, 2, 5, 10\} \), \( n_5(G) \in \{1, 6\} \), \( n_{10}(G) \in \{1, 3\} \)

\( n_3(G) \in \{1, 6\} \) but \( G \) is simple so \( n_3(G) \neq 1 \), \( n_5(G) \neq 6 \)

The 3- and 5-sylow subgps are cyclic of order 3, 5 respectively \( (30 = 2 \cdot 3 \cdot 5) \), so the 3's and 5's are separately disjoint.

So \( G \) has \( 10 \cdot 2 = 20 \) elements of order 3

and \( 6 \cdot 4 = 24 \) " " " 5.

But \( G \) doesn't have \( 44 > 30 \) distinct elements, so \( G \) doesn't exist.