Math 322, lecture 12, 19/10/2014

Last time: Action of $G$ on $X = \text{map} : G \times X \to X$

$\iff$ Homomorphism $G \to S_X$.

**Question 2:** Give action to $g \in G$ we associated $\sigma_g : X \to X$ by

$\sigma_g(x) = g \cdot x$.

Saw:
1. $\sigma_g \in S_X$
2. map $g \mapsto \sigma_g$ is a hom $G \to S_X$

**Analogue:** To $f \in C(C[0,1])$, set $(f^\prime)(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

facts: $(\alpha f + g)^\prime = \alpha f^\prime + g^\prime$

$\Rightarrow$ map $D : C(C[0,1]) \to C(C[0,1])$ given by $Df = f^\prime$

is linear.

**Today:** Conjugation

This will be an action of $G$ on itself, but not the regular action.

**Def:** For $g \in G$, set $g x = \sigma_g(x) = g \cdot x \cdot g^{-1}$.

**Lemma:** (1) This is an action of $G$ on itself

(2) It's an action by automorphisms: $\sigma_g \in \text{Aut}(G)$

(3) $\tau : G \to \text{Aut}(G)$ is a hom

**Ex.** Need to show: $g^h x = g(h \cdot x)$, $e_x = x$. 
(2) \( \delta_g(xy) = \delta_g(x) \delta_g(y) \)

(3) \( \gamma : G \to S_G \) is a hom, and we checked in (2) that its image is contained in \( \text{Aut}(G) \)

**Def:** Say \( x, y \in G \) are **conjugate** if there is \( g \in G \) s.t. \( y = gxg^{-1} \).

**Lemma:** This is an equivalence relation

**Pf.:** PS3, Problem 2(a)

**Def:** The equivalence classes are **called** the **Conjugacy classes** in \( G \). Sometimes the set of classes is denoted \( G^h \).

**Example:**
- Conjugacy class of \( e \) is \( \{ e \} \); \( ge g^{-1} = e \) for all \( g \in G \).
- Conjugacy class of \( g \) is \( \{ g^k \} \) iff \( g \in \text{Z}(G) \)
  \( (gxg^{-1} = x \iff gx = xg) \)

**Remark:** Conjugacy is useful because (1) acts by **automorphisms**

(2) **readily available**: no need for anything beyond \( G \).

**Def:** Image of \( \gamma : G \to \text{Aut}(G) \) is called the **group of**
inner automorphisms, denoted $\text{Inn}(G)$

Ex: $\text{Ker}(\gamma) = Z(G)$ so $\text{Inn}(G) \leq G/Z(G)$.

Also, if $f \in \text{Aut}(G)$, $g \in G$ then $f \circ g \circ f^{-1} = f(g)$

So $\text{Inn}(G)$ is normal in $\text{Aut}(G)$

Def: $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$, called "outer automorphism group of $G$".

Examples: $G = \mathbb{Z}^d$ conjugacy is trivial since $G$ is commutative: $g + x + (-g) = x$ for all $g, x$.

So $\text{Inn}(\mathbb{Z}^d) = \{ \text{id}_{\mathbb{Z}^d} \}$ but $\text{Aut}(\mathbb{Z}^d) = \text{GL}_d(\mathbb{Z})$

Example: If $\#X \geq 3$, then $\mathbb{Z}(S_X) = \{ \text{id} \}$

So $\text{Inn}(S_X) = S_X$

Fact: $\text{Out}(S_n) = \{ e \}$ except $\text{Out}(S_6) \cong C_2$

If $f: S_n \to S_n$ (n\neq 6)

is an isom the, $f(\sigma) = \phi \circ \sigma \circ \phi^{-1}$

Lemma: There is a bijection between the conjugacy class of $x \in G$ and the coset space $G/Z_G(x)$. 
Cor: The number of conjugates of \( x \) is \( |G : \mathbb{Z}_G(x)| \)

\[ \text{Pf:} \quad \text{Map} \quad g \mathbb{Z}_G(x) \rightarrow g \mathbb{Z}_G(x) \quad \mathbb{Z}_G(x) \quad g \cdot x = gxg^{-1}. \]

(1) well-defined: if \( g' = g, \forall \mathbb{Z}_G(x) \text{ then } g'x = g'x(g')^{-1} = (g2) \times (g7)^{-1} = g(2x7^{-1})g^{-1} = gxg^{-1} = gx \]

\[ 7 \in \mathbb{Z}_G(x) \]

(1) surjective: \( g \times g^{-1} \) is image of \( g \mathbb{Z}_G(x) \)

(2) injective: if \( g'x = g'x \), then \( g'g'x = g'g''x = e \), so \( g'g' \times g^{-1} = g''x(g'y') \)

\[ x = g^{-1}g' \times (g')^{-1} = (g^{-1}g')(g^{-1}g') \]

so \( g^{-1}g' \in \mathbb{Z}_G(x) \), so \( g \mathbb{Z}_G(x) = g' \mathbb{Z}_G(x) \)

Thm: (Class equation) Let \( G \) be finite. Then

\[ \#G = \#\mathbb{Z}(G) + \sum_{x \neq x} [G : \mathbb{Z}_G(x)] \]

where the sum is over the non-central conjugacy classes.

\[ \text{Pf: } G \text{ is the only disjoint union of its conjugacy classes.} \]

Note: Every summand divides \( \#G \).
Ex. Interpret class equation as a combinatorial identity when $G = S_5$

Review: Conjugacy of subgroups

Def: For $g \in G$, $H \leq G$ set $gHg^{-1} = N_g(H)$

Lemma: This is an action of $G$ on its set of subgroups.

Pf: Same as before: $eH = eHe^{-1} = H$

and $g^hH = (gh)H(gh)^{-1} = ghHh^{-1}g^{-1} = g(g^{-1}Hg^{-1})g' = g(hH)$

Def: Call $H, K \leq G$ conjugate if $gH = K$ for some $g \in G$

Lemma: This is an equivalence relation.

Ex. Example: The class of $H$ is $|H|$ if $H$ is normal.

Lemma: There is a bijection between class of $H$ and $G/\text{N}_G(H)$

Pf: Same as map $g \text{N}_G(H) \mapsto gHg^{-1}$

(b) Well-def: if $n \in \text{N}_G(H)$, then $(gn)H(gn)^{-1} = g(hHh^{-1})g^{-1} = gHg^{-1}$

(i) Surjective: $gHg^{-1}$ is image of $g\text{N}_G(H)$

(ii) Injective: if $gHg^{-1} = hHh^{-1}$ then $(g^{-1}h)H(g^{-1}h)^{-1} = H$

so $g^{-1}h \in \text{N}_G(H)$ and $g\text{N}_G(H) = h\text{N}_G(H)$
Second Review: General actions

Fix group $G$ acting on a set $X$.

**Def:** Say $x, y \in X$ are in the same orbit if $\exists g \in G: gx = y$.

**Lemma:** This is an equivalence relation.

*Proof:* $e \cdot x = x$ so $x, x$ are in same orbit.

If $g \cdot x = y$ then $x = g^{-1}(gy) = g^{-1}y$ so $y, x$ are in same orbit.

If $y = g \cdot x$, $\tau = h \cdot y$ then $\tau = h \cdot (g \cdot x) = (hg) \cdot x$.

**Def:** Equivalence classes are called the orbit of $G$ on $X$.

Write $G \cdot x$ or $O(x)$ for the orbit of $x \in X$.

Write $G \backslash X$ for the set of orbits.

**Example:** (Poincare): $X =$ space of solar system point $x \in X =$ specifying positions and velocities of all planets.

Action $(\mathbb{R}, +)$ on $X$: $g_t \cdot x = y$ where if we start system at point $x$ at time $0$, it reaches $y$ at time $t$. 
