Nilpotent groups

1. Fix a field $F$ (say $F = \mathbb{R}$) and $n \geq 2$. Let $U_n \subset \text{GL}_n(F)$ denote the group of upper-triangular matrices with $1$s on the main diagonal. Write $E^{ij}$ for the matrix having $1$ at position $ij$ and $0$ everywhere else.

(a) Show that $Z(U_n) = U_{n,n-1} = \{ I_n + zE^{1,n} \mid z \in F \}$, the matrices whose only non-zero entry (above the main diagonal) is in the upper right corner.

(b) Show that the equivalence class of $u \in U_n$ in $U_n/Z(U_n)$ depends exactly on the entries of $u_n$ except the corner one.

(c) Show that $U_{n,n-1} = \{ u \in U_n \mid 2 \leq j < i + n - 2 \rightarrow u_{ij} = 0 \}$ is the subgroup $Z^2(U_n) \triangleleft U_n$ which contains $Z(U_n)$ and such that $Z^2(U_n)/Z(U_n)$ is the center of $U_n/Z(U_n)$.

(d) For each $1 \leq m \leq n-1$ let

$$U_{n,m} = \{ u \in U_n \mid 2 \leq j < i + m \rightarrow u_{ij} = 0 \} = \left\{ I_n + \sum_{j-i \geq m} z_{ij}E^{ij} \mid z_{ij} \in F \right\}$$

be the group with non-zero entries starting in the $m$th diagonal above the main diagonal (note that $U_{n,1} = U_n$). Show that $U_{n,m}$ is normal in $U_n$.

(e) Show that $U_{n,m}/U_{n,m+1}$ is the center of $U_n/U_{n,m+1}$ and conclude that $Z^i(U_n) = U_{n,n-i}$ and that $U_n$ is nilpotent.

**Definition.** For $A, B \subset G$ write $[A, B]$ for the subgroup $\langle \{ [a, b] \mid a \in A, b \in B \} \rangle$ generated by all commutators of elements from $A, B$.

2. (Descending central series) Let $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.

(a) Show by induction that $\gamma_i(G)$ are normal subgroups such that $\gamma_{i+1}(G) \subset \gamma_i(G)$.

(b) Show that $\gamma_i(G)/\gamma_{i+1}(G)$ is contained in the centre of $G/\gamma_{i+1}(G)$.

(c) Suppose $G$ is nilpotent of degree $d$, so that $Z^d(G) = G$. Show that $\gamma_i(G) \subset Z^{d+1-i}(G)$.

Solvable groups

3. Let $G$ be a group of order $p^aq^b$. In each case show that $G$ is solvable (hint: find a normal subgroup $N$ and consider $N$ and $G/N$ separately).

(a) $a = 2, b = 1$.

(b) $a = 2, b = 2$.

4. Let $n \geq 2$ and let $B_n(F) \subset \text{GL}_n(F)$ be the subgroup of upper-triangular invertible matrices.

(a) Show that $U_n \triangleleft B_n$ and that $B_n/U_n \cong (F^\times)^n$.

(b) Show that $B_n(F)$ is solvable.

(c) Show that (unless $F = \mathbb{F}_2$) $Z(B_n(F))$ consists exactly of the scalar matrices with non-zero entries.

(d) Show that for a large enough field $F$, $Z(B_n/Z(B_n)) = \{ e \}$. Conclude that $B_n$ is solvable but not nilpotent (this holds for any $F \neq \mathbb{F}_2$).
The derived series

5. Fix a group $G$. The subgroup $G' = [G, G]$ is called the derived subgroup.
   (a) Show that $G'$ is a normal subgroup of $G$.
   (b) Let $N < G$. Show that $G/N$ is abelian iff $G' \subset N$.

   DEF The descending series of subgroups defined by $G^{(0)} = G$ and $G^{(i+1)} = \left( G^{(i)} \right)'$ is called the derived series.
   (c) Show that $G^{(i)}/G^{(i+1)}$ is abelian.

6. Let $G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_k$ be a descending series of subgroups of $G$ with $G_{i-1}/G_i$ abelian.
   Note that we don’t assume $G_k = \{e\}$.
   (a) Writing $G^{(1)} = G'$ show that $G^{(1)} \subset G_1$.
   (b) Writing $G^{(2)} = (G')'$ show that $G^{(2)} \subset G'_1 \subset G_2$.
   (c) Writing $G^{(i+1)} = \left( G^{(i)} \right)'$ show by induction that $G^{(i)} \subset G_i$ for each $i$.
   (d) Show that $G$ is solvable iff $G^{(n)} = \{e\}$ for some $n$.

Supplementary problems

Let $G$ be a nilpotent group.
A. Show that $X$ generates $G$ iff its image generates $G^{\text{ab}} = G/G'$.
B. Suppose $G$ is finitely generated. Show that every subgroup of $G$ is finitely generated.
B. Show that $G_{\text{tors}}$ is a subgroup of $G$. 