Lior Silberman’s Math 322: Problem Set 6 (due 24/10/2017)

Practice problems

P1. Let $G$ be a group and let $X$ be a set of size at least 2. Fix $x_0 \in X$ and for $g \in G, x \in X$ set $g \cdot x = x_0$.
   (a) Show that this operation satisfies $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.
   (b) This is not a group action. Why?

P2. Let $G$ act on $X$. Say that $A \subset X$ is $G$-invariant if for every $g \in G, a \in A$ we have $g \cdot a \in A$.
   (a) Show that this operation satisfies $g \cdot (ah) = (g \cdot a)h$ for all $g, h \in G, a \in A$.
   (b) Suppose $A$ is $G$-invariant iff $g \cdot A = A$ ($g \cdot A$ in the sense of problem 4(a)).
   (c) Show that $A$ is $G$-invariant. Show that the restriction of the action to $A$ (formally, the binary operation $\cdot |_{G \times A}$) is an action of $G$ on $A$.

Simplicity of $A_n$

1. Let $V = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$. Show that $V \triangleleft S_4$, so that $S_4$ is not simple.

2. (The normal subgroups of $S_n$) Let $N \triangleleft S_n$ with $n \geq 5$.
   (a) Let $G$ be a group and let $H < G$ be a normal subgroup isomorphic to $C_2$. Show that $H < Z(G)$.
   (b) Suppose that $N \cap A_n \neq \{\text{id}\}$. Show that $N \triangleright A_n$ and conclude that $N = A_n$ or $N = S_n$.
   (c) Suppose that $N \cap A_n = \{\text{id}\}$. Show that $N$ is isomorphic to a subgroup of $C_2$.
   (d) Show that if $n \geq 3$ then $Z(S_n) = \{\text{id}\}$, and conclude that in case (c) we must have $N = \text{id}$.

3. Let $X$ be an infinite set.
   (a) Show that $S^\text{fin}_X = \{\sigma \in S_X \mid \text{supp}(\sigma) \text{ is finite}\}$ is a subgroup of $S_X$.
   PRAC For finite $F \subset X$ there is a natural inclusion $S_F \hookrightarrow S_X$, which is a group homomorphism and an isomorphism onto its image. Let $\text{sgn}_F : S_F \rightarrow \{\pm 1\}$ be the sign character.
   DEF For $\sigma \in S^\text{fin}_X$ define $\text{sgn}(\sigma) = \text{sgn}_F(\sigma)$ for any finite $F$ such that $\sigma \in S_F$.
   (c) Show that $\text{sgn}(\sigma)$ is well-defined (independent of $F$) and a group homomorphism $S^\text{fin}_X \rightarrow \{\pm 1\}$.
   (*d) The infinite alternating group $A_X$ is kernel of this homomorphism. Show that $A_X$ is simple.

Group actions

4. Let the group $G$ act on the set $X$.
   (a) For $g \in G$ and $A \in P(X)$ set $g \cdot A = \{g \cdot a \mid a \in A\} = \{x \in X \mid \exists a \in A : x = g \cdot a\}$. Show that this defines an action of $G$ on $P(X)$.
   (b) In PS2 we endowed $P(X)$ with a group structure. Show that the action of (a) is by automorphisms: that the map $A \mapsto g \cdot A$ is a group homomorphism $(P(X), \Delta) \rightarrow (P(X), \Delta)$.
   (c) Let $Y$ be another set. For $f : X \rightarrow Y$ set $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Show that this defines an action of $G$ on $Y^X$, the set of functions from $X$ to $Y$.
   (*d) Suppose that $Y = \mathbb{R}$ (or any field), so that $\mathbb{R}^X$ has the structure of a vector space over $\mathbb{R}$. Show that the action of (c) is by linear maps.
5. (Some stabilizers) The action of $S_X$ on $X$ induces an action on $P(X)$ as in problem 4(a). Suppose that $X$ is finite, $\#X = n$.

(a) Let $A, B \subset X$. Show there is $\sigma \in S_X$ such that $\sigma \cdot A = B$ iff $\#A = \#B$ (we’ll call $\binom{X}{k} = \{A \subset X \mid \#A = k\}$ an orbit of the action of $S_X$ on $P(X)$).

SUPP When $X$ is infinite, $\binom{X}{k}$ are orbits if $\kappa < |X|$, but there are multiple orbits on $\binom{X}{|X|}$, parametrized by the cardinality of the complement.

(b) Let $A \subset X$. Show that $\text{Stab}_{S_X}(A) \overset{\text{def}}{=} \{\sigma \in S_X \mid \sigma \cdot A = A\} \simeq S_A \times S_{X-A}$ (we call $\text{Stab}_{S_X}(A)$ the “stabilizer” of $A$).

(c) Compute the index $[S_X : \text{Stab}_{S_X}(A)]$. Now read Proposition 175 in the notes and use it to show that $\#\binom{X}{k} = \frac{n!}{k!(n-k)!}$.

Supplementary problem: Conjugation

A. Let $G$ be a finite group.

(a) Suppose all elements of $G$ are conjugate. Show that $G = \{e\}$.

(b) Suppose $G$ has exactly two conjugacy classes. Show that $G \simeq C_2$.

(**c) Suppose $G$ has exactly three conjugacy classes. Show that $G \simeq C_3$ or $G \simeq S_3$.

RMK There exists an infinite group in which all non-identity elements are conjugate.

**B. Show that for each $k$ there is $N = N(k)$ such that every finite group with $k$ conjugacy classes has order at most $N$.

(hint for 2(a): let $H = \{1, h\}$, let $g \in G$, and consider the element $ghg^{-1}$)

(hint for 2(b): consider the index of $N$)

(hint for 2(c): restrict sgn: $S_n \to C_2$ to $N$)

(hint for 6(b): the number of conjugates of an element divides the order of the group)