Practice Problems

P1 Let $G$ be a group with $|G| = 2$. Show that $G = \{e, g\}$ with $g \cdot g = e$ (hint: consider the multiplication table). Show that $G \simeq C_2$ (that is, find an isomorphism $C_2 \to G$).

P2 Let $G$ be a group. Give a bijection between $\{H < G \mid \#H = 2\}$ and $\{g \in G \mid g^2 = e, g \neq e\}$.

P3. Are these groups? In each case either prove the group axioms or show that an axiom fails.
(a) The non-negative real numbers with the operation $x \ast y = \max \{x, y\}$.
(b) $\mathbb{R} \setminus \{-1\}$ with the operation $x \ast y = x + y + xy$.

P4 (Basics of groups and homomorphisms) Fix groups $G, H, K$ and let $f \in \text{Hom}(G, H)$.
(a) Given also $g \in \text{Hom}(H, K)$, show that $g \circ f \in \text{Hom}(G, K)$.
(b) Suppose $f$ is bijective. Then $f^{-1} : H \to G$ is a homomorphism.

Groups and Homomorphisms

1. Let $G$ be a group, and let $(A, +)$ be an abelian group. For $f, g \in \text{Hom}(G, A)$ and $x \in G$ define $(f + g)(x) = f(x) + g(x)$ (on the right this is addition in $A$).
(a) Show that $f + g \in \text{Hom}(G, A)$.
(b) Show that $(\text{Hom}(G, A), +)$ is an abelian group.
(*c) Let $G$ be a group, and let $\text{id} : G \to G$ be the identity homomorphism. Define $f : G \to G$ by $f(x) = (\text{id}(x))(\text{id}(x)) = x \cdot x = x^2$. Suppose that $f \in \text{Hom}(G, G)$. Show that $G$ is commutative.

2. (External Direct products) Let $G, H$ be groups.
(a) On the product set $G \times H$ define an operation by $(g, h) \cdot (g', h') = (g g', h h')$. Show that $(G \times H, \cdot)$ is a group.
DEF this is called the (external) direct product of $G, H$.
(b) Let $\tilde{G} = \{(g, e_H) \mid g \in G\}$ and $\tilde{H} = \{(e_G, h) \mid h \in H\}$. Show that $\tilde{G}, \tilde{H}$ are subgroups of $G \times H$ and that $\tilde{G} \cap \tilde{H} = \{e_G \times e_H\}$.
SUPP Show that $\tilde{G}, \tilde{H}$ are isomorphic to $G, H$ respectively.
(c) Show that for any $x = (g, h) \in G \times H$ we have $x \tilde{G} x^{-1} = \tilde{G}$ and $x \tilde{H} x^{-1} = \tilde{H}$ (the notation means $x \tilde{G} x^{-1} = \{gx^{-1} \mid g \in \tilde{G}\}$).
EXAMPLE The Chinese remainder theorem shows that $C_n \times C_m \simeq C_{nm}$ if gcd $(n, m) = 1$.

3. Products with more than two factors can be defined recursively, or as sets of $k$-tuples.
SUPP Find “natural” isomorphisms $G \times H \simeq H \times G$ and $(G \times H) \times K \simeq G \times (H \times K)$. We therefore write products without regard to the order of the factors.
DEF Write $G^k$ for the $k$-fold product of groups isomorphic to $G$.
(a) Show that every non-identity element of $C_2^k$ has order 2.
(b) Show that $C_3 \times C_3 \ncong C_9$.

4. The Klein group or the four-group is the group $V \simeq C_2 \times C_2$.
PRAC Check that $(\mathbb{Z}/12\mathbb{Z})^\times \simeq V$ and that $(\mathbb{Z}/8\mathbb{Z})^\times \simeq V$.
(a) Write a multiplication table for $V$, and show that $V$ is not isomorphic to $C_4$.
(b) Show that $V = H_1 \cup H_2 \cup H_3$ where $H_i \subset V$ are subgroups isomorphic to $C_2$. 

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(c) Let $G$ be a group of order 4. Show that $G$ is isomorphic to either $C_4$ or to $C_2 \times C_2$.

5. Let $G$ be a group, and let $H, K < G$ be subgroups and suppose that $H \cup K$ is a subgroup as well. Show that $H \subset K$ or $K \subset H$. 
Extra credit

6. Show that, for each \( d \mid n \), \( \mathbb{Z}/n\mathbb{Z} \) has a unique subgroup of order (=size) \( d \) (and that the subgroup is cyclic).

7**. Let \( G \) be a finite group of order \( n \), and suppose that for each \( d \mid n \) \( G \) has at most one subgroup of order \( d \). Show that \( G \) is cyclic.

Supplementary Problems

A. Let \( G \) be the isometry group of the Euclidean plane: \( G = \{ f : \mathbb{R}^n \to \mathbb{R}^n \mid \| f(x) - f(y) \| = \| x - y \| \} \).
(a) Show that every \( f \in G \) is a bijection and that \( G \) is closed under composition and inverse.
(b) For \( a \in \mathbb{R}^n \) set \( t_a(x) = x + a \). Show that \( t_a \in G \), and that \( a \mapsto t_a \) is an injective group homomorphism \((\mathbb{R}^n, +) \to G\).

DEF Call the image the subgroup of translations and denote it by \( T \).
(c) Let \( K = \{ g \in G \mid g(0) = 0 \} \). Show that \( K \leq G \) is a subgroup (we usually denote it \( O(n) \) and call it the orthogonal group).

DEF This is called the orthogonal group and consists of rotations and reflections.

FACT \( K \) acts on \( \mathbb{R}^n \) by linear maps.
(d) Show \( \forall g \in G \exists t \in T : g0 = t0 \), and hence that \( t^{-1}g \in K \). Conclude that \( G = TK \).
(e) Show that every \( x \in G \) has a unique representation in the form \( g = tk, t \in T, k \in K \) (hint: what is \( T \cap K \)?)
(f) Show that \( K \) normalizes \( T \): if \( k \in K, t \in T \) we have \( ktk^{-1} \in T \) (hint: use the linearity of \( k \)).
(g) Show that \( T \triangleleft G \): for every \( g \in G \) we have \( gTg^{-1} = T \).

RMK We have shown that \( G \) is the semidirect product \( G = K \rtimes T \).

B. Let \( X \) be a set of size at least 2, and fix \( e \in X \). Define \( *: X \times X \to X \) by \( x * y = y \).
(a) Show that \( * \) is an associative operation and that \( e \) is a left identity.
(b) Show that every \( x \in X \) has a right inverse: an element \( \bar{x} \) such that \( x * \bar{x} = e \).
(c) Show that \( (X, *) \) is not a group.

C. Let \( \{ G_i \}_{i \in I} \) be groups. On the cartesian product \( \prod_i G_i \) define an operation by

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(g \cdot h)_i = g_ih_i
\]

(that is, by co-ordinatewise multiplication).
(a) Show that \( (\prod_i G_i, \cdot) \) is a group.

DEF This is called the (external) direct product of the \( G_i \).
(b) Let \( \pi_j : \prod_i G_i \to G_j \) be projection on the \( j \)th coordinate. Show that \( \pi_j \in \text{Hom}(\prod_i G_i, G_j) \).
(c) (Universal property) Let \( H \) be any group, and suppose given for each \( i \) a homomorphism \( f_i \in \text{Hom}(H, G_i) \). Show that there is a unique homomorphism \( f : H \to \prod_i G_i \) such that for all \( i \), \( \pi_i \circ f = f_i \).

(**d) An abstract direct product of the groups \( G_i \) is a pair \( (G, \{ q_i \}_{i \in I}) \) where \( G \) is a group, \( q_i : G \to G_i \) are homomorphisms, and the property of (c) holds. Suppose that \( G, G' \) are both abstract direct products of the same family \( \{ G_i \}_{i \in I} \). Show that \( G, G' \) are isomorphic (hint: the system \( \{ q_i \} \) and the universal property of \( G' \) give a map \( \varphi : G \to G' \), and the same idea gives a map \( \psi : G' \to G \). To see that the composition is the identity compare for example \( q_i \circ \psi \circ \varphi, q_i \circ \text{id}_G \) and use the uniqueness of (c).
D. Let $V, W$ be two vector spaces over a field $F$. On the set of pairs $V \times W = \{(v, w) \mid v \in V, w \in W\}$ define $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $a \cdot (v_1, w_1) = (a \cdot v_1, a \cdot w_1)$.

(a) Show that this endows $V \times W$ with the structure of a vector space. This is called the external direct sum of $V, W$ and denote it $V \oplus W$.

(b) Generalize the construction to an infinite family of vector spaces as in problem C(a).

(*c) State a universal property analogous to that of C(c), C(d) and prove the analogous results.

E. (Supplement to P3) Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Then $f : [0, 2\pi) \to S^1$ given by $f(\theta) = (\cos \theta, \sin \theta)$ is continuous, 1-1 and onto but its inverse is not continuous.