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Abstract. In this note we prove that the Hasse-Weil zeta function of a curve is a rational function and satisfies a functional equation. We follow [Must, Chapter 3].

1. Preliminaries and notation: quick review

Throughout this note $X$ is a smooth projective curve over $k := \mathbb{F}_q$. A Weil divisor $D \in \text{Div}(X)$ on $X$ is a finite formal sum of the form

$$D = \sum_{x \in X_{\text{cl}}} n_x x,$$

where $X_{\text{cl}}$ are the closed points of $X$. We identify each closed point in $X_{\text{cl}}$ with the orbit of a point in $X(\mathbb{F}_q)$ under the action of $\text{Gal}(\mathbb{F}_q|\mathbb{F}_q)$. The degree of a closed point $x \in X_{\text{cl}}$ is $\deg(x) = [k(x) : k]$, where $k(x)$ is the residue field of $x$. The degree of the divisor $D$ is

$$\deg(D) = \sum_{x \in X_{\text{cl}}} n_x \deg(x).$$

Example 1.1. Let $X = \mathbb{A}^1_{\mathbb{F}_3} = \text{spec}(\mathbb{F}_3[x])$. Then $P = \text{spec}\left(\mathbb{F}_3[x]/(x^2 + 1)\right)$ is a closed point of $X$ corresponding to the maximal ideal $(x^2 + 1)$ of $\mathbb{F}_3[x]$. The residue field is $\mathbb{F}_3(P) = \mathbb{F}_3[x]/(x^2 + 1)$ which is a degree 2 extension of $\mathbb{F}_3$. Hence the divisor $D = P$ has degree $\deg(D) = [\mathbb{F}_3(P) : \mathbb{F}_3] = 2$.

Because $X$ is a smooth projective curve we may identify a Weil divisor $D \in \text{Div}(X)$ with its induced line bundle $L = \mathcal{O}_X(D)$. We write $\deg(\mathcal{O}_X(D)) = \deg(D)$.

We say that two Weil divisors $D, D' \in \text{Div}(X)$ are linearly equivalent and write $D \sim D'$ iff $D - D' = \text{div}(f)$ for some $f \in k(X)^\times$. We write Pic$(X)$ to denote the group of the divisors on $X$ modulo this equivalence relation. Note that linearly equivalent Weil divisors correspond to isomorphic line bundles. In other words, Pic$(X)$ is the group of line bundles on $X$ modulo the isomorphism relation. We write $[D]$ for a divisor class in Pic$(X)$.

Since our curve $X$ is projective, linearly equivalent divisors have the same degree and hence the degree map descends to give a group homomorphism $\deg : \text{Pic}(X) \to \mathbb{Z}$. The kernel of this homomorphism is denoted by Pic$^0(X)$. We recall the Riemann-Roch theorem.

Theorem 1.2. Let $D \in \text{Div}(X)$ and write $\mathcal{K}$ for the canonical divisor of $X$. We have

$$\ell(D) - \ell(\mathcal{K} - D) = \deg D - g + 1.$$
Moreover, \( \deg(K) = 2g - 2 \) and
\[
\ell(D) = \deg(D) - g + 1, \text{ if } \deg(D) \geq 2g - 1.
\]

In the following we will make use of the following corollary of the Riemann-Roch.

**Proposition 1.3.** The number of effective divisors in \( \text{Div}(X) \) that are linearly equivalent to \( D \in \text{Div}(X) \) is \( \frac{q^{\ell(D)} - 1}{q - 1} \). If in particular \( \deg(D) \geq 2g - 1 \), then the number of effective divisors in \( \text{Div}(X) \) that are linearly equivalent to \( D \) is \( \frac{q^{\deg(D) - g + 1} - 1}{q - 1} \).

**Remark 1.4.** Recall that for \( D, D' \in \text{Div}(X) \) with \( D \sim D' \) we have \( \ell(D) = \ell(D') \). Therefore the integer \( \ell([D]) := \ell(D) \) is well defined for a divisor class in \( \text{Pic}(X) \).

2. **Rationality**

In this section we aim to prove the following strong form of the rationality conjecture in the setting of a smooth projective curve \( X \) over \( \mathbb{F}_q \).

In the following we write \( \text{Pic}_0(X) = \{[D] \in \text{Pic}(X) : \deg([D]) = 0 \} \), and
\[
\text{Pic}_m(X) = \{[D] \in \text{Pic}(X) : \deg([D]) = m \},
\]
To state the strong form of the rationality conjecture we aim to prove, we will first see that \( \text{Pic}_0(X) \) is a finite subgroup of \( \text{Pic}(X) \). We will write \( h := |\text{Pic}_0(X)| \).

**Lemma 2.1.** We have that

1. \( \text{Pic}_0(X) \) is a finite subgroup of \( \text{Pic}(X) \), we write \( h := |\text{Pic}_0(X)| \).
2. \( \deg(\text{Pic}(X)) = e\mathbb{Z} \) for some \( e \in \mathbb{N}_{>0} \) and if we write \( h := |\text{Pic}_0(X)| \), we have
\[
|\{[D] \in \text{Pic}(X) : \deg([D]) = m \}| = \begin{cases} h, & e|m \\ 0, & \text{otherwise} \end{cases}
\]

**Proof.** We will first prove the first part of this lemma. It is easy to see that \( \text{Pic}_0(X) \) is a group. We will prove that \( \text{Pic}_0(X) \) is finite. Let \( D_n \in \text{Div}(X) \) be such that \( \deg(D_n) := n \geq 2g \). Notice that the map
\[
\text{Pic}_0(X) \to \text{Pic}_n(X)
\]
\[
[D] \mapsto [D + D_n],
\]
gives a bijection between \( \text{Pic}_0(X) \) and \( \text{Pic}_n(X) \). Therefore, it suffices to prove that \( \text{Pic}_n(X) \) is a finite set. We claim that for any divisor class \( [D] \in \text{Pic}_n(X) \), there exists an effective divisor \( D' \in \text{Div}(X) \) such that \( [D] = [D'] \). This is a consequence of the Riemann-Roch. Since \( \deg(D) \geq 2g > 2g - 1 \), we have that \( \ell(D) = n - g + 1 > 0 \), hence \( D \) is linearly equivalent to an effective divisor \( D' \). Thus it suffices to see that there is a finite number of effective divisors of degree \( n \). This holds since there are only finitely many ways to write
n as a sum of positive numbers and there are only finitely many closed points in $X_{\text{cl}}$ with degree less than $n$.

For the second part of this lemma, notice that $\deg(\text{Pic}(X))$ is an ideal of $\mathbb{Z}$, therefore it can be written as $\deg(\text{Pic}(X)) = e\mathbb{Z}$ for some $e \in \mathbb{N}_{>0}$. Fix a divisor class $[D_m] \in \text{Pic}^{\text{em}}(X)$. The map

$$\text{Pic}^0(X) \to \text{Pic}^{\text{em}}(X)$$

$$[D] \mapsto [D + D_m],$$

gives a bijection between $\text{Pic}^0(X)$ and $\text{Pic}^{\text{em}}(X)$. The lemma follows. □

**Theorem 2.2.** If $X$ is a smooth projective curve over $\mathbb{F}_q$ of genus $g$ such that $X$ is irreducible over $\mathbb{F}_q$, we have

$$Z(X, t) = \frac{f(t)}{(1-t)(1-qt)},$$

where $f \in \mathbb{Z}[t]$ is a polynomial of degree $\deg(f) \leq 2g$, such that $f(0) = 1$ and $f(1) = h$.

We begin by establishing some key lemmas.

**Lemma 2.3.** Let $X$ be a variety over $\mathbb{F}_q$ and $X'$ be the same variety over $\mathbb{F}_q$. Then

$$Z(X', t') = \prod_{i=1}^r Z(X, \xi^i t'),$$

where $\xi$ is a primitive $r$–th root of unity.

**Proof.** Let $N_m = |X(\mathbb{F}_q^m)|$ and $N'_m = |X'(\mathbb{F}_q^m)|$. We want to prove that

$$\exp \left( \sum_{m \geq 1} \frac{N'_m}{m} t^m \right) = \prod_{i=1}^r \exp \left( \sum_{\ell \geq 1} \frac{N_{\ell i}}{\ell} \xi^{\ell i} t^\ell \right),$$

or equivalently that

$$\sum_{m \geq 1} \frac{N'_m}{m} t^m = \sum_{\ell \geq 1} \frac{N_{\ell}}{\ell} \left( \sum_{i=1}^r \xi^{\ell i} \right) t^\ell.$$

The desired equality follows from the fact that $N'_m = N_{rm}$ for all $m \geq 1$ and

$$\sum_{i=1}^r \xi^{\ell i} = \begin{cases} 0 & \text{if } r \nmid \ell \\ r & \text{otherwise.} \end{cases}$$

□

**Proof of Theorem 2.2.** Last time we saw that

$$Z(X, t) = \sum_{D \geq 0} t^{\deg(D)}.$$
Denote by $a_{[D]} := |\{D' \in [D] : D' \geq 0\}|$. We may write

$$Z(X, t) = \sum_{[D] \in \text{Pic}(X)} a_{[D]} t^{\deg([D])}.$$  

We break this sum into two components depending on whether $\deg([D]) \geq 2g - 1$ or $\deg([D]) \leq 2g - 2$. Then

$$Z(X, t) = \sum_{[D] \in \text{Pic}(X), \deg([D]) \leq 2g - 2} a_{[D]} t^{\deg([D])} + \sum_{[D] \in \text{Pic}(X), \deg([D]) \geq 2g - 1} a_{[D]} t^{\deg([D])}. \tag{1}$$

We will now prove the first part of this theorem. That is that $Z(X, t)$ is a rational function. Notice that $S_1(t) := \sum_{[D] \in \text{Pic}(X), \deg([D]) \leq 2g - 2} a_{[D]} t^{\deg([D])}$, is a polynomial. Therefore, it suffices to prove that $S_2(t) := \sum_{[D] \in \text{Pic}(X), \deg([D]) \geq 2g - 1} a_{[D]} t^{\deg([D])}$ is a rational function. By Proposition 1.3, we get

$$S_2(t) = \sum_{[D] \in \text{Pic}(X), \deg([D]) \geq 2g - 1} \frac{q^{\deg([D]) - g + 1} - 1}{q - 1} t^{\deg([D])}. \tag{2}$$

Notice now that in view of Lemma 2.1 we have $\deg(\text{Pic}(X)) = e\mathbb{Z}$ for some $e \in \mathbb{N} > 0$ and

$$|\{[D] \in \text{Pic}(X) : \deg([D]) = m\}| = \begin{cases} h & e|m \\ 0 & \text{otherwise}. \end{cases}$$

Let $d_0$ be the smallest integer such that $d_0e \geq 2g - 1$. We have

$$S_2(t) = \sum_{d \geq d_0} h \frac{q^{d - g + 1} - 1}{q - 1} t^d = \frac{h}{(q - 1)} \left( q^{1-g} \cdot \frac{(qt)^{d_0e}}{1 - (qt)^e} - \frac{t^{d_0e}}{1 - t^e} \right). \tag{3}$$

This finishes the proof that $Z(X, t)$ is a rational function.

We proceed now to establish the complete statement of the theorem. Notice that $S_1(t) = g(t^e)$ for some polynomial $g \in \mathbb{Z}[t]$ with degree $\deg(g) \leq \frac{2g - 2}{e}$. Combining this with (3) we get

$$Z(X, t) = \frac{f(t^e)}{(1 - t^e)(1 - q^e t^e)}, \tag{4}$$

where $f \in \mathbb{Q}[t]$ with $\deg(f) \leq \max\{2 + \frac{2g - 2}{e}, d_0 + 1\}$. In fact since $Z(X, t) \in \mathbb{Z}[[t]]$ we see that $f \in \mathbb{Z}[t]$. We will now show that $e = 1$. 

Note that the fact that $S_1(t)$ is a polynomial together with the expression in (3) yield

$$\lim_{t \to 1} (t - 1)Z(X, t) = \lim_{t \to 1} \frac{-ht^{d_0e}(t - 1)}{q(1 - te)}.$$  

Therefore, $Z(X, t)$ has a pole of order 1 at $t = 1$. If we now consider $X'$ to be the curve $X$ over $F_{q^e}$, in view of Lemma 2.3, we get

$$Z(X', t^e) = \prod_{i=1}^{e} Z(X, \xi^i t),$$

for a $e$–th primitive root of unity $\xi$. This equation combined with (4) gives that

$$Z(X', t^e) = Z(X, t)^e.$$  

However as we have seen $Z(X, t)$ as well as $Z(X', t)$ has a pole of order 1 at $t = 1$, which gives $e = 1$.

We have thus established that $e = 1$.

Since $e = 1$, we have $d_0 = 2g - 1$. Therefore, the fact that $\deg(f) \leq \max\{2 + \frac{2g-2}{e}, d_0 + 1\}$ implies that $\deg(f) \leq 2g$.

If in particular $g = 0$ we have

$$Z(X, t) = \frac{h}{(1 - t)(1 - qt)}.$$  

Finally, if $g \geq 1$ we have $f(0) = 1$ and $f(1) = h$ as one can easily see from (2) and (3). \hfill \Box

In the course of the proof of Theorem 2.2 we saw that $\deg(\text{Pic}(X)) = \mathbb{Z}$. Combining this with Lemma 2.1, we get the following corollary.

**Corollary 2.4.** All $\text{Pic}^m(X)$ have the same non-zero number of elements $h = |\text{Pic}^0(X)|$.

Before proceeding to prove the functional equation, we make some remarks on the existence of divisors with degree one.

**Remark 2.5.**

- If a curve $X$ has an $F_q$ point, then it has a divisor over $F_q$ of degree 1. However, the converse is not true.
- If a curve defined over any field $K$, has genus 0 or 1 then it has an $K$–point if and only if it has a divisor over $K$ of degree 1. This is a consequence of the Riemann-Roch which in this case implies that every divisor is linearly equivalent to an effective divisor.
- Corollary 2.4 is not true for smooth curves over a number field $K$. For example we consider the smooth conic $C : x^2 + y^2 + z^2 = 0$ over $\mathbb{Q}$. This conic has no divisor of degree 1. Indeed, if it had, since it has genus 0 it would also have a $\mathbb{Q}$–point. However $C$ is pointless over $\mathbb{Q}$. In fact, one can see that since the canonical divisor of $C$ has degree $-2$ we have $\deg(\text{Pic}(C)) = 2\mathbb{Z}$. 

• If the curve $X$ has a divisor of degree $n$ and $m$ for two coprime integers $n$ and $m$, then it has a divisor of degree 1. Most times (more specifically when the genus of the curve is not 1) we can find a divisor of even degree. This is the case because the canonical divisor on curve of genus $g$ has degree $2g - 2$. Thus if a curve of genus not equal to one has a divisor of odd degree, then it also has a divisor of degree 1.

3. Functional equation

We are now going to prove that the Hasse-Weil zeta function of a curve satisfies a functional equation, as stated in the theorem below.

**Theorem 3.1.** If $X$ is a smooth projective curve over $\mathbb{F}_q$ of genus $g$ such that $X$ is irreducible over $\overline{\mathbb{F}_q}$, we have

$$Z \left( X, \frac{1}{qt} \right) = q^{1-g} t^{2g} Z(X, t).$$

**Proof.** As in the proof of Theorem 2.2 we write

$$Z(X, t) = \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^{|D|} - 1}{q - 1} t^m + \sum_{m \geq 2g-1} \frac{h m^{g-1} - 1}{q - 1} t^m$$

$$= \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^{|D|} - 1}{q - 1} t^m + \frac{h}{(q-1)} (q^{1-g} \cdot (qt)^{2g-1} - \frac{t^{2g-1}}{1-t})$$

$$= \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^{|D|}}{q - 1} t^m + \frac{h}{(q-1)} (q^{1-g} \cdot (qt)^{2g-1} - \frac{1}{1-t}) := F(t) + G(t),$$

where $F(t) = \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^{|D|}}{q - 1} t^m$ and $G(t) = \frac{h}{(q-1)} (q^{1-g} \cdot (qt)^{2g-1} - \frac{1}{1-t}).$

We now compute

$$\frac{(q-1)}{h} G(1/qt) = q^{1-g} \cdot \frac{t^{1-2g}}{1-t^{-1}} - \frac{1}{1-(qt)^{-1}}$$

$$= \frac{q^{1-g} t^{2g}}{t-1} - \frac{qt}{qt-1}$$

$$= t^{2-2g} q^{1-g} \left( \frac{qt^{2g-1}}{1-qt} - \frac{1}{t-1} \right)$$

$$= t^{2-2g} q^{1-g} \frac{1}{h} G(t).$$
Therefore,
\[(5) \quad G(1/qt) = t^{2-2g}q^{1-g}G(t).\]

Next are going to compute \(F(1/qt)\). We have
\[
F(1/qt) = \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^\ell([D])}{q-1} (qt)^m.
\]

In view of Theorem 1.2, we have that the map
\[
\text{Pic}^m(X) \to \text{Pic}^{2g-2-m} \quad [D] \mapsto [\mathcal{K} - D],
\]
is a bijection. Moreover, as \(m\) runs through \(\{0, \cdots, 2g-2\}\) so does \(2g-2-m\). Thus, the sum can be rewritten as
\[
F(1/qt) = \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^\ell([\mathcal{K} - D])}{q-1} (qt)^{m+2-2g}.
\]
Furthermore, Theorem 1.2 yields that \(\ell([\mathcal{K} - D]) = \ell([D]) - (\deg([D]) - g + 1)\). Therefore, we get
\[
F(1/qt) = \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^\ell([D]) - m + g - 1}{q-1} (qt)^{m+2-2g}
= t^{2-2g}q^{1-g} \sum_{m=0}^{2g-2} \sum_{[D] \in \text{Pic}^m(X)} \frac{q^\ell([D])}{q-1} t^m
= t^{2-2g}q^{1-g} F(t).
\]
Thus,
\[(6) \quad F(1/qt) = t^{2-2g}q^{1-g} F(t).\]
Combining (5) and (6) we get
\[
Z(X, 1/qt) = q^{1-g} t^{2-2g} Z(X, t).
\]
The theorem follows. \(\square\)

References

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