The exponential

1. Products of absolutely convergent series.
   (a) Let $V$ be a normed space, and let $T, S \in \text{End}_b(V)$ commute. Show that $\exp(T + S) = \exp(T) \exp(S)$.

   (b) Show that, for appropriate values of $t$, $\exp(A) \exp(B) \neq \exp(A + B)$ where $A = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}$.

Companion matrices

DEF The companion matrix associated with the polynomial $p(x) = x^n - \sum_{i=0}^{n-1} a_i x^i$ is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}.$$ 

2. A sequence $\{x_k\}_{k=0}^{\infty}$ is said to satisfy a linear recurrence relation if for each $k$,

$$x_{k+n} = \sum_{i=0}^{n-1} a_i x_{k+i}.$$ 

(a) Define vectors $\mathbf{v}^{(k)} = (x_{k-n+1}, x_{k-n+2}, \ldots, x_k)$. Show that $\mathbf{v}^{(k+1)} = C \mathbf{v}^{(k)}$ where $C$ is the companion matrix.

(b) Find $x_{100}$ if $x_0 = 1$, $x_1 = 2$, $x_2 = 3$ and $x_n = x_{n-1} + x_{n-2} - x_{n-3}$.

PRAC Find the Jordan canonical form of $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

3. Let $C$ be the companion matrix associated with the polynomial $p(x) = x^n - \sum_{k=0}^{n-1} a_k x^k$.

(a) Show that $p(x)$ is the characteristic polynomial of $C$.

(b) Show that $p(x)$ is also the minimal polynomial.

(c) Find (with proof) an eigenvector with eigenvalue $\lambda$.

(**d) Let $g$ be a polynomial, and let $\mathbf{v}$ be the vector with entries $v_k = \lambda^k g(k)$ for $0 \leq k \leq n-1$.

Show that, if the degree of $g$ is small enough (depending on $p, \lambda$), then $((C - \lambda) \mathbf{v})_k = \lambda \left( g(k+1) - g(k) \right) \lambda^k$ and (the hard part) that $((C - \lambda) \mathbf{v})_{n-1} = \lambda \left( g(n) - g(n-1) \right) \lambda^{n-1}$.

(**e) Find the Jordan canonical form of $C$. 

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**Holomorphic calculus**

Let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) be a power series with radius of convergence \( R \). For a matrix \( A \) define \( f(A) = \sum_{m=0}^{\infty} a_m A^m \) if the series converges absolutely in some matrix norm.

5. Let \( D = \text{diag} (\lambda_1, \cdots, \lambda_n) \) be diagonal with \( \rho(D) < R \) (that is, \(|\lambda_i| < R\) for each \( i \)). Show that \( f(D) = \text{diag} (f(\lambda_1), \cdots, f(\lambda_n)) \).

6. Let \( A \in M_n(\mathbb{C}) \) be a matrix with \( \rho(A) < R \).
   
   (a) [review of power series] Let \( R' \) satisfy \( \rho(A) < R' < R \). Show that \( |a_m| \leq C (R')^{-m} \) for some \( C > 0 \).
   
   (b) Using PS8 problem 3(a) show that \( f(A) \) converges absolutely with respect to any matrix norm.

   (*c) Suppose that \( A = S (D + N) S^{-1} \) where \( D + N \) is the Jordan form (\( D \) is diagonal, \( N \) upper-triangular nilpotent). Show that
   
   \[ f(A) = S \left( \sum_{k=0}^{n} \frac{f^{(k)}(D)}{k!} N^k \right) S^{-1}. \]

   **Hint:** \( D, N \) commute.
   
   **RMK1** This gives an alternative proof that \( f(A) \) converges absolutely if \( \rho(A) < R \), using the fact that \( f^{(k)}(D) \) can be analyzed using single-variable methods.
   
   **RMK2** Compare your answer with the Taylor expansion \( f(x + y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} y^k \).

   (d) Apply this formula to find \( \exp(tB) \) where \( B \) is as in PS9 problem 2.

7. Let \( A \in M_n(\mathbb{C}) \). Prove that \( \det(\exp(A)) = \exp(\text{Tr}A) \).

**Supplementary problems**

A. Let \( p \in \mathbb{C}[x] \) be a polynomial, let \( D' \) be the derivative operator for distributions in \( C^\infty_c(\mathbb{R})' \). Show that \( \varphi \in C^\infty_c(\mathbb{R})' \) satisfies \( p(D') \varphi = 0 \) iff \( \varphi \) is given by integration against a function \( f \) such that \( p(D) f = 0 \).