Lior Silberman’s Math 412: Problem Set 6 (due 26/10/2016)

P1. (Minimal polynomials)
Let \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}. \)

(a) Find the minimal polynomial of \( A \) and show that the minimal polynomial of \( B \) is \( x^2(x-1)^2 \).
(b) Find a \( 3 \times 3 \) matrix whose minimal polynomial is \( x^2 \).

P2. For each of \( A, B \) find its eigenvalues and the corresponding generalized eigenspaces.

**Triangular matrices**

P3. Let \( L \) be a lower-triangular square matrix with non-zero diagonal entries. Find a formula for its inverse.

1. Let \( U \) be an upper-triangular square matrix with non-zero diagonal entries.
   (a) Give a “backward-substitution” algorithm for solving \( Ux = b \) efficiently.
   (b) Explicitly use your algorithm to solve \( \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 3 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \).
   (c) Give a formula for \( U^{-1} \), proving in particular that \( U \) is invertible and that \( U^{-1} \) is again upper-triangular.

RMK We’ll see that if \( \mathcal{A} \subset M_n(F) \) is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in \( \mathcal{A} \) belongs to \( \mathcal{A} \). This applies, in particular, to the set of upper-triangular matrices.

**The minimal polynomial**

2. Let \( D \in M_n(F) = \text{diag}(a_1, \ldots, a_n) \) be diagonal.
   (a) For any polynomial \( p \in F[x] \) show that \( p(D) = \text{diag}(p(a_1), \ldots, p(a_n)) \).
   (b) Show that the minimal polynomial of \( D \) is \( m_D(x) = \prod_{j=1}^r (x-a_{i_j}) \) where \( \{a_{i_j}\}_{j=1}^r \) is an enumeration of the distinct values among the \( a_i \).
   (c) Show that (over any field) the matrix \( B \) from problem P1 is not similar to a diagonal matrix.
   (d) Now suppose that \( U \) is an upper-triangular matrix with diagonal \( D \). Show that for any \( p \in F[x], p(U) \) has diagonal \( p(D) \). In particular, \( m_D|_{m_U} \).

3. Let \( T \in \text{End}(V) \) be diagonalizable. Show that every generalized eigenspace is simply an eigenspace.

4. Let \( S \in \text{End}(U), T \in \text{End}(V) \). Let \( S \oplus T \in \text{End}(U \oplus V) \) be the “block-diagonal map”.
   (a) For \( f \in F[x] \) show that \( f(S \oplus T) = f(S) \oplus f(T) \).
   (b) Show that \( m_{S \oplus T} = \text{lcm}(m_S, m_T) \) (“least common multiple”: the polynomial of smallest degree which is a multiple of both).
   (c) Conclude that \( \text{Spec}_F(S \oplus T) = \text{Spec}_F(S) \cup \text{Spec}_F(T) \).

RMK See also problem B below.
5. Let $R \in \text{End}(U \oplus V)$ be “block-upper-triangular”, in that $R(U) \subset U$.
   (a) Define a “quotient linear map” $\tilde{R} \in \text{End}(U \oplus V/U)$.
   (b) Let $S$ be the restriction of $R$ to $U$. Show that both $m_S$, $m_R$ divide $m_R$.
   (c) Let $f = \text{lcm}[m_S, m_R]$ and set $T = f(R)$. Show that $T(U) = \{0\}$ and that $T(V) \subset U$.
   (d) Show that $T^2 = 0$ and conclude that $f | m_R | f^2$.
   (e) Show that $\text{Spec}_F(R) = \text{Spec}_F(S) \cup \text{Spec}_F(\tilde{R})$.

### Supplementary problems

**A. (Cholesky decomposition)**

(a) Let $A$ be a positive-definite square matrix. Show that $A = LL^\dagger$ for a unique lower-triangular matrix $L$ with positive entries on the diagonal.

**DEF** For $\varepsilon \in \pm 1$ define $D_\varepsilon \in M_n(\mathbb{R})$ by $D_{ij}^\varepsilon = \begin{cases} \varepsilon & j = i + \varepsilon \\ -\varepsilon & j = i \\ 0 & j \neq i, i + \varepsilon \end{cases}$

(positive) discrete Laplace operator.

(b) To $f \in C^\infty(0, 1)$ associate the vector $\frac{1}{n}D_{+}f$ and $\frac{1}{n}D_{-}f$ are both close to $f''$ (so that both are discrete differentiation operators). Show that $\frac{1}{n^2}D_{-}D_{+}$ is an approximation to the second derivative.

(c) Find a lower-triangular matrix $L$ such that $LL^\dagger = A$.

**B.** Let $T \in \text{End}(V)$. For monic irreducible $p \in \mathbb{F}[x]$ define $V_p = \{ v \in V \mid \exists k : p(T)^k v = 0 \}$.

(a) Show that $V_p$ is a $T$-invariant subspace of $V$ and that $m_T|_{V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{0\}$. Conclude that $p^k|_{m_T}$.

(b) Show that if $\{p_i\}_{i=1}^V \subset \mathbb{F}[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^V V_{p_i}$ is direct.

(c) Let $\{p_i\}_{i=1}^V \subset \mathbb{F}[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^V V_{p_i}$.

(d) Suppose that $m_T(x) = \prod_{i=1}^V p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \text{Ker} p_i^{k_i}(T)$.

**C. (more on extension of scalars)** Let $F \subset K$ be fields and let $V$ be an $F$-vectorspace. Let $V_K = K \otimes_F V$ thought of as a $K$-vectorspace.

(a) (Repeat of supplement to Problem 1 of PS5) For $T \in \text{Hom}_F(U, V)$ let $T_K = \text{Id}_K \otimes_F T \in \text{Hom}_F(U_K, V_K)$ be the tensor product map. Show that $T_K \in \text{Hom}_K(U_K, V_K)$ (that is, $K$ linear and not only $F$-linear).

(b) Let $\{u_j\}_{j \in J} \subset U$, $\{v_i\}_{i \in I} \subset V$ be a bases. Show that the matrix of $T_K$ wrt the bases $\{1 \otimes u_j\}_{j \in J} \subset U_K$, $\{1 \otimes v_i\}_{i \in I} \subset V_K$ is the matrix of $T$ wrt $\{u_j\}_{j \in J}$, $\{v_i\}_{i \in I}$.

(c) Show that the minimal and characteristic polynomials of $T_K$ are those of $T$ (through the inclusion of $F$ in $K$).