(1) If the series converges, find its sum. Otherwise, state that it diverges.

(a) \( \sum_{n=0}^{\infty} \left( -\frac{3}{11} \right)^n \)

Solution: We rewrite the series as
\[
\sum_{n=0}^{\infty} 3^3 \left( -\frac{3}{11} \right)^n = 27 \sum_{n=0}^{\infty} \left( -\frac{9}{11} \right)^n
\]
we now see that we have a convergent geometric series, which sums to
\[
= 27 \cdot \frac{1}{1 - \left( -\frac{9}{11} \right)} = \frac{27 \cdot 11}{20}.
\]

(b) \( \sum_{n=1}^{\infty} \left( -\frac{3}{11} \right)^n \)

Solution: We rewrite the series as
\[
\sum_{n=1}^{\infty} 3^2 \left( -\frac{3}{11} \right)^n = 9 \sum_{n=1}^{\infty} \left( -\frac{27}{11} \right)^n.
\]
This is a divergent geometric series (its ratio \(-\frac{27}{11}\) has magnitude greater than 1).

(2) Decide whether the following series converge:

(a) \( \sum_{n=0}^{\infty} \frac{n}{2^n} \)

Solution: We have \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2^{n+1}} / \frac{n}{2^n} = \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2} \left( 1 + \frac{1}{n} \right) \xrightarrow[n \to \infty]{} \frac{1}{2} < 1 \) so the series converges by the ratio test.

(b) \( \sum_{n=0}^{\infty} \frac{n!}{2^n} \)

Solution: We have \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{2^{n+1}} / \frac{n!}{2^n} = \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} = \frac{n+1}{2} \xrightarrow[n \to \infty]{} \infty > 1 \) so the series diverges by the ratio test.

(c) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \)

Solution: We have \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{n+1} \xrightarrow[n \to \infty]{} 0 < 1 \) so the series converges by the ratio test.

(d) For which values of \( x \) does \( \sum_{n=0}^{\infty} nx^n \) converge?

Solution: Let \( a_n = nx^n \). Then
\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1) |x|^{n+1}}{n |x|^n} = \left( 1 + \frac{1}{n} \right) |x| \xrightarrow[n \to \infty]{} |x|.
\]
By the ratio test, the series converges if \( |x| < 1 \) and diverges if \( |x| > 1 \). If \( |x| = 1 \) then \( |a_n| = n \cdot |x|^n = n \xrightarrow[n \to \infty]{} \infty \) so the series diverges by the divergence test. We conclude that the series converges exactly when \( |x| < 1 \), that is for \( x \in (-1, 1) \).