Math 101 – SOLUTIONS TO WORKSHEET 28

ABSOLUTE CONVERGENCE

1. MORE TAIL ESTIMATES

(1) It is known that
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2. \]
How many terms are needed for the error to be less than 0.01?

Solution: The series is alternating, so the error in approximating its sum by a partial sum is less than the first omitted term. Taking the first 99 terms, this means that
\[ \left| \log 2 - \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} \right) \right| \leq \frac{1}{100} \]
as desired.

(2) It is known that
\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}. \]
How many terms are needed for the error to be less than 0.001?

Solution: Again the series is alternating. The magnitude of the nth term is \( \frac{1}{2n-1} \) so taking the first 500 terms we get that
\[ \left| \frac{\pi}{4} - \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{999} \right) \right| \leq \frac{1}{1001} < \frac{1}{1000}. \]

2. CONVERGENCE

(3) Which of the following converges:

□ \( \left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty} \)

□ \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

□ \( \left\{ \frac{(-1)^n}{\sqrt{n}} \right\}_{n=1}^{\infty} \)

□ \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \)

Solution: \( \lim_{n \to 1} \frac{1}{\sqrt{n}} = 0 \), so also \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \), and by the squeeze theorem \( \lim_{n \to \infty} \frac{(-1)^n}{\sqrt{n}} = 0 \), so both sequences converge. The series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is a p-series with \( p = \frac{1}{2} < 1 \) so it diverges while the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges by the alternating series test.

(4) Place checkmarks

<table>
<thead>
<tr>
<th>( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} )</th>
<th>Converges</th>
<th>Diverges</th>
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<tbody>
<tr>
<td>( \sum_{n=1}^{\infty} \frac{1}{n^2} )</td>
<td>Absolutely</td>
<td>Conditionally</td>
</tr>
<tr>
<td>( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} )</td>
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</tr>
<tr>
<td>( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} )</td>
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3. RATIO TEST

(5) Decide whether the following series converge:

(a) \( \sum_{n=0}^{\infty} \frac{n}{2^n} \)

Solution: We have \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2} \cdot \left( 1 + \frac{1}{n} \right) \to \frac{1}{2} < 1 \) so the series converges by the ratio test.

Date: 16/3/2016, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.
(b) \( \sum_{n=0}^{\infty} \frac{n!}{2^n} \)

**Solution:** We have
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!/2^{n+1}}{n!/2^n} \right| = \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} = \frac{n+1}{2} \xrightarrow{n \to \infty} \infty > 1
\]
so the series diverges by the ratio test.

(c) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \)

**Solution:** We have
\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{n+1} \xrightarrow{n \to \infty} 0 < 1
\]
so the series converges by the ratio test.

(d) For which values of \( x \) does \( \sum_{n=0}^{\infty} nx^n \) converge?

**Solution:** Let \( a_n = nx^n \). Then
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)x^{n+1}}{n x^n} \right| = \left( 1 + \frac{1}{n} \right) |x| \xrightarrow{n \to \infty} |x|.
\]
By the ratio test, the series *converges* if \( |x| < 1 \) and *diverges* if \( |x| > 1 \). If \( |x| = 1 \) then
\[
|a_n| = n |x|^n = n \xrightarrow{n \to \infty} \infty
\]
so the series *diverges* by the divergence test. We conclude that the series converges exactly when \( |x| < 1 \), that is for \( x \in (-1,1) \).