Math 101 – SOLUTIONS TO WORKSHEET 17
APPROXIMATE INTEGRATION

1. APPROXIMATE INTEGRATION

(1) Let \( f(x) = \sin(x^2) \). Estimate \( \int_0^1 f(x) \, dx \) using the trapezoid rule, the midpoint rule, and Simpson’s rule, with \( n = 4 \) in all cases. You may leave your answers in calculator-ready form.

**Solution:** With \( n = 4 \) we have \( \Delta x = \frac{1}{4} \) and the points \( 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \), so the approximations are:

\[
\int_0^1 f(x) \, dx \approx \frac{1}{8} \left( \sin(0^2) + 2\sin \left( \frac{1}{4} \right)^2 + 2\sin \left( \frac{1}{2} \right)^2 + 2\sin \left( \frac{3}{4} \right)^2 + \sin(1^2) \right)
\]

\[
= \frac{1}{8} \left( 2\sin \left( \frac{1}{16} \right) + 2\sin \left( \frac{1}{4} \right) + 2\sin \left( \frac{9}{16} \right) + \sin(1) \right),
\]

\[
\int_0^1 f(x) \, dx \approx \frac{1}{4} \left( \sin \left( \frac{1}{8} \right)^2 + \sin \left( \frac{9}{64} \right) + \sin \left( \frac{25}{64} \right) + \sin \left( \frac{49}{64} \right) \right)
\]

and

\[
\int_0^1 f(x) \, dx \approx \frac{1}{12} \left( \sin(0) + 4\sin \left( \frac{1}{16} \right) + 2\sin \left( \frac{1}{4} \right) + 4\sin \left( \frac{9}{16} \right) + \sin(1) \right)
\]

\[
= \frac{1}{12} \left( 4\sin \left( \frac{1}{16} \right) + 2\sin \left( \frac{1}{4} \right) + 4\sin \left( \frac{9}{16} \right) + \sin(1) \right).
\]

(2) (Final 2009) Give the Simpson’s rule approximation to \( \int_0^2 \sin(e^x) \, dx \) using 4 equal subintervals.

**Solution:** Here \( \Delta x = \frac{2}{4} = \frac{1}{2} \), the points are \( 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \) and so the approximation is

\[
\frac{1}{6} \left( \sin(e^0) + 4\sin \left( e^{1/2} \right) + 2\sin(1) + 4\sin \left( e^{3/2} \right) + \sin(e^2) \right)
\]

which is

\[
\frac{1}{6} \left( \sin(1) + 4\sin \left( e^{1/2} \right) + 2\sin(e) + 4\sin \left( e^{3/2} \right) + \sin(e^2) \right).
\]

(3) (Final 2012) Let \( I = \int_{1/2}^1 \frac{x}{x} \, dx \).

(a) Write down Simpson’s rule approximation for \( I \) using 4 points (call it \( S_4 \))

**Solution:** \( S_4 = \frac{1}{12} \left( \frac{1}{1} + 4\pi/4 + 2\pi/2 + 4\pi/4 + \frac{1}{2} \right) \).

It was not required to do the arithmetic, but for the record we note (since \( 210 = 2 \cdot 3 \cdot 5 \cdot 7 \)):

\[
S_4 = \frac{1}{12} \left( 1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2} \right)
= \frac{1}{12} \left( 210 + 42 \cdot 16 + 70 \cdot 3 + 30 \cdot 16 + 105 \right)
= \frac{1677}{2520}.
\]

(b) Without computing \( I \), find an upper bound for \( |I - S_4| \). You may use the fact that if \( |f^{(4)}(x)| \leq K \) on \( [a, b] \) then the error in the approximation with \( n \) points has magnitude at most \( K(b - a)^5/180n^4 \).
We have \( f'(x) = -\frac{1}{x^2} \), \( f''(x) = \frac{2}{x^3} \), \( f'''(x) = -\frac{6}{x^4} \) and \( f^{(4)}(x) = \frac{24}{x^5} \). On the interval \([1, 2]\), the function \( \frac{24}{x^5} \) is decreasing so \( |f^{(4)}(x)| \leq \frac{24}{x^5} = 24 \). It follows that the error is at most
\[
\frac{24(2-1)^5}{180 \cdot 4^4} = \frac{24}{180 \cdot 256} = \frac{1}{60 \cdot 32} = \frac{1}{1920}.
\]

(4) (Final 2008) Let \( I = \int_0^1 \cos(x^2) \, dx \). It can be shown that the fourth derivative of \( \cos(x^2) \) has absolute value less than or equal to 0.001. You may use that that if \( |f^{(4)}(t)| \leq K \) for \( a \leq t \leq b \) then error in using Simpson’s rule to approximate \( \int_a^b f(x) \, dx \) has absolute value less than or equal to \( K(b-a)^3/180n^4 \).

**Solution:** For \( f(x) = \cos(x^2) \) we are given that \( |f^{(4)}(x)| \leq 60 \) for \( 1 \leq x \leq 2 \), so we need \( n \) such that
\[
\frac{60 \cdot (1-0)^5}{180n^4} \leq \frac{1}{1000},
\]
which is the same as
\[
n^4 \geq \frac{1000}{3}.
\]
Now for \( n = 6 \) we have \( 6^4 = 36 \cdot 6 \geq 30 \cdot 30 = 900 > \frac{1000}{3} \) so \( n = 6 \) suffices.

(5) Let \( I = \int_4^6 \sin(\sqrt{x}) \, dx \). Find \( n \) such that estimating \( I \) using the midpoint rule and \( n \) points will have an error of at most \( \frac{1}{1000} \). You may use that the absolute error in estimating \( \int_a^b f(x) \, dx \) using the midpoint rule and \( n \) points is at most \( K(b-a)^3/24n^2 \) where \( |f^{(4)}(x)| \leq K \) for \( a \leq x \leq b \).

**Solution:** Let \( f(x) = \sin(\sqrt{x}) \). Then \( f'(x) = \frac{1}{2\sqrt{x}} \cos(\sqrt{x}) \) so \( f''(x) = \frac{1}{4x^{3/2}} \cos(\sqrt{x}) - \frac{1}{4x} \sin(\sqrt{x}) \). For \( 4 \leq x \leq 6 \) we have \( \frac{1}{4x^{3/2}} \leq \frac{1}{4 \cdot 4^{3/2}} = \frac{1}{32} \) \( (\frac{1}{x^{3/2}} \) is decreasing on this interval) and \( \frac{1}{4x} \leq \frac{1}{4 \cdot 4} = \frac{1}{16} \) (for the same reason). Since \( |\cos(\sqrt{x})|, |\sin(\sqrt{x})| \leq 1 \) for all \( x \), we have
\[
|f''(x)| \leq \frac{1}{32} + \frac{1}{16} = \frac{3}{32} \leq \frac{3}{30} = \frac{1}{10}
\]
for all \( 4 \leq x \leq 6 \). It follows that the error in the approximation is at most
\[
\frac{1}{10} \cdot \frac{(6-4)^3}{24 \cdot n^2} = \frac{8}{240n^2} = \frac{1}{30n^2}.
\]
For \( n = 10 \) the error would be at most \( \frac{1}{30000} = \frac{1}{3000} \) so that is enough.