Math 101, Lecture 35 (3/4/2017)

Review Session 1

(1) Limits without l'Hôpital

(2) Review

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(3) Let $g(x) = \begin{cases} \frac{e^{-x^2}-1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) Find $g^{(3)}(0)$.

- Expand in MacLaurin series:
- Divide by x
- Read of derivative from Coeff

$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ so $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

So $e^{-x^2} - 1 = -x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

So $\frac{e^{-x^2}-1}{x} = -x + \frac{x^3}{2} - \frac{x^5}{6} + \dots$

Coeff of x^3 is $\frac{g^{(3)}(0)}{3!}$ so $\frac{g^{(3)}(0)}{3!} = \frac{1}{2}$

so $g^{(3)}(0) = \frac{6}{2} = 3.$

(b) (2011 Final) Give the first three non-zero terms of the MacLaurin series for $\int g(x)\,dx$.

- Integrate term-by-term
(don't forget C.)

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2. Limits without l'Hôpital's rule

(4) (Final 2012) Evaluate $\lim_{x\to 0} \frac{\sin(x)-x+x^3/6}{\sin(x^5)}$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \ldots$ ← recall expansion of sin x

so $\sin x - x + \frac{x^3}{6} = \frac{x^5}{120} +$ ~~lower~~ higher order

and $\sin(x^5) = x^5 +$ ~~lower~~ higher order

$\Rightarrow \frac{\sin(x) - x + x^3/6}{\sin(x^5)} = \frac{\frac{x^5}{120} + \text{small}}{x^5 + \text{small}} =$

$= \frac{\frac{1}{120} + \text{higher order}}{1 + \text{higher order}} \xrightarrow[x\to 0]{} \frac{1/120}{1} = \frac{1}{120}$

(5) Evaluate $\lim_{x\to 0} \frac{x\sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)}$

notice cancellation only to 3rd order, 4th order terms don't cancel, so work to 4th order

$x\sin x = x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^8}{7!} + \ldots$

$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} + \ldots \Rightarrow \log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \ldots$

$e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + \ldots, \quad \cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$

so $\frac{x\sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)} = \frac{\left(x^2 - \frac{x^4}{6} + \ldots\right) - \left(x^2 - \frac{x^4}{2} + \ldots\right)}{\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right)} =$

$= \frac{\frac{1}{3}x^4 + \ldots}{\frac{2}{24}x^4 + \ldots} = \frac{\frac{1}{3} + \text{higher order}}{\frac{1}{12} + \text{higher order}} \xrightarrow[x\to 0]{} \frac{1/3}{1/12} = 4.$

Question: When expanding f into a power series, does the domain change?

Yes - domain of convergence can be smaller

Example: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ← defined if $-1<x<1$

↑ defined if $x \neq 1$

Question: Find Taylor expansion of $\sin(x)$ about $c = \pi/3$

Solution: let $f(x) = \sin(x)$ $f(c) = \frac{\sqrt{3}}{2}$

then $f'(x) = \cos(x)$ so $f'(c) = \frac{1}{2}$

$f''(x) = -\sin(x)$ $f^{(2)}(c) = -\frac{\sqrt{3}}{2}$

$f^{(3)}(x) = -\cos(x)$ $f^{(3)}(c) = -\frac{1}{2}$

$f^{(4)}(x) = \sin(x)$

⋮ ← contine periodically

So Taylor expansion is $\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{2!}\left(\frac{\sqrt{3}}{2}\right)\left(x-\pi/3\right)^2 - \frac{1}{3!}\left(\frac{1}{2}\right)\left(x-\frac{\pi}{3}\right)^3$

$+ \frac{1}{4!}\left(\frac{\sqrt{3}}{2}\right)\left(x-\pi/3\right)^4 + \frac{1}{5!}\left(\frac{1}{2}\right)\left(x-\pi/3\right)^5 - \frac{1}{6!}\left(\frac{\sqrt{3}}{2}\right)\left(x-\pi/3\right)^6 - \cdots$

(expansion of f is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$)

Alternative: $\sin(x) = \sin\left(\left(x-\pi/3\right) + \pi/3\right) = \sin\left(x-\pi/3\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(x-\pi/3\right)\sin\left(\pi/3\right)$

Find Expansion of $\int \frac{e^x-1}{x}\,dx$ about $x=0$. (to 4th order)

Know $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$

so $e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$

so $\frac{e^x-1}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots$

so $\int \frac{e^x-1}{x}\,dx = C + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \cdots$

Or:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$, so $e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

so $\frac{e^x-1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ so $\int \frac{e^x-1}{x}\,dx = C + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!}\,dx =$

↑ integrate term-by-term

$= C + \sum_{n=1}^{\infty} \frac{1}{n\cdot(n!)} \cdot x^n$