34. Taylor Series: Derivatives and Limits (31/3/2017)

Goals:

(1) Reading derivatives off power series.
(2) Taking limits using Taylor series

Where are we? ⊛ Represented function as power series

(a) Using (massaging) known expansions

(b) Using Taylor formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

⊛ Discussed convergence of such expansions (ratio test for radius, other test on boundary)

⊛ Applications: (a) Approximating functions: $\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

(b) Getting derivatives & integrals $\log 2 = \sum_{n=1}^{\infty} \frac{1}{n 2^n}$

Question: How do we tell if $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges absolutely?

(ratio test proves absolute convergence, but $\lim_{n\to\infty} \frac{1}{(n+1)^5} / \frac{1}{n^5} = 1$)

(absolute convergence means $\sum_{n=1}^{\infty} |a_n|$ converges, here $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^5}\right| = \sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series)) solutions

("terms decay like $\frac{1}{n^5}$ which is fast enough for convergence, but slower than exponential decay") thoughts to have

Can use Taylor's formula "forward" - given f, find series.

Can use it "backwards": given series, find $f^{(n)}(c)$

1. Derivatives

(1) (Final 2014) Let $\sum_{n=0}^{\infty} c_n x^n$ be the MacLaurin series for e^{3x}. Find c_5. (and $\frac{d^5}{dx^5}(e^{3x})\big|_{x=0}$)

Recall: $e^u = \sum_{n=0}^{\infty} \frac{1}{n!} u^n$

so $e^{3x} = \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot x^n$, so $c_5 = \frac{3^5}{5!} = \frac{f^{(5)}(0)}{5!}$

so if $f(x) = e^{3x}$, $f^{(5)}(0) = 3^5 = 243$

(Can also differentiate 5 times, find $f^{(5)}(0)$, say $c_5 = \frac{f^{(5)}(0)}{5!}$)

(2) (Final 2013) Let $f(x) = x^2 \sin(x^3)$. Find $f^{(11)}(0)$.

$\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u^{2n+1}$

$x^2 \cdot (x^3)^{2n+1} = x^{6n+5}$

so $f(x) = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (x^3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{6n+5}$

Know coeff of x^{11} is $\frac{f^{(11)}(0)}{11!}$ so $\frac{f^{(11)}(0)}{11!} = \frac{(-1)^1}{3!} = -\frac{1}{3!}$

term with x^{11} has $n=1$

so $f^{(11)}(0) = -\frac{11!}{3!}$

Date: 31/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

What is $f^{(12)}(0)$? No x^{12} term, so $\frac{f^{(12)}(0)}{12!} = 0$ so $f^{(12)}(0) = 0$

If term is missing, its coeff is 0.

(3) Let $g(x) = \begin{cases} \frac{e^{-x^2}-1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) Find $g^{(3)}(0)$.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

So $e^{-x^2} - 1 = -x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$

So $\frac{e^{-x^2} - 1}{x} = -x + \frac{x^3}{2!} - \frac{x^5}{6} + \frac{x^7}{24} - \cdots$

So $\frac{g^{(3)}(0)}{3!} = \frac{1}{2}$ so $g^{(3)}(0) = \frac{3!}{2} = \frac{6}{2} = 3$

Coeff of x^3 by Taylor's formula

(b) (2011 Final) Give the first three non-zero terms of the MacLaurin series for $\int g(x)\,dx$.

$$\int \left(-x + \frac{x^3}{2} - \frac{x^5}{6} + \cdots\right) dx = C - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{36} + \cdots$$

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2. Limits without l'Hôpital's rule

(4) (Final 2012) Evaluate $\lim_{x\to 0} \frac{\sin(x)-x+x^3/6}{\sin(x^5)}$

Idea: Expand numerator + denominator into power series, divide:

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ so $\sin(x) - x + \frac{x^3}{6} = \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(x^5) = x^5 - \frac{x^{15}}{3!} + \dots$

so $\frac{\sin(x)-x+x^3/6}{\sin(x^5)} = \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5 - \frac{x^{15}}{3!} + \dots} = \frac{\frac{1}{5!} - \frac{x^2}{7!} + \dots}{1 - x^{10}/3! + \dots} \xrightarrow[x\to 0]{} \frac{1/5!}{1} = \frac{1}{5!}$

divide both by x^5

(5) Evaluate $\lim_{x\to 0} \frac{x\sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)}$

Remark: It is ok to plug in 0 to a series