33. Taylor SEries Manipulating (30/3/2017)

Goals:

(1) Expanding functions into Taylor series.
(2) Reading derivatives off power series.

Last time: Getting new expansions from old ones by diff., int.

e.g. $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\Rightarrow$ (integrate) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. (const $= 0$ since $\arctan(0) = 0$)

Non need to change index in derivatives / integral / etc

Enough to say: $\frac{x^3}{1-x} = \sum_{n=0}^{\infty} x^{n+3}$

$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} n x^{n-1}$

Finishing last lecture: Summing series using known expansions

Worksheet 1, 2, 3

Skill: recognize a series as a special case / derivative / integral of a known expansion

Short Table of standard expansions

You must either memorize the following expansions or be able to quickly reproduce them.

- (geometric series)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- (Exponential)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- (Trig)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

- (logarithm)

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

- (inverse tangent)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

1. Manipulating power series: summing series

(1) Find $\sum_{n=1}^{\infty} \frac{1}{n2^n}$. ("looks like logarithm expansion!")

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(-\frac{1}{2}\right)^n \cdot (-1) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(-\frac{1}{2}\right)^n$$

$$\underset{\text{logarithm series}}{=} -\log\left(1+\left(-\frac{1}{2}\right)\right) = -\log\left(\frac{1}{2}\right) = \log 2$$

, radius of convergence is 1, and $\left|\frac{1}{2}\right| < 1$

(2) Avatars of geometric series.

(a) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Try $\sum_{n=1}^{\infty} nx^n$ instead (plug in $x=\frac{1}{2}$ later). Note: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Differentiate both sides to get: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}$

multiplying by x, get $\sum_{n=1}^{\infty} nx^n = x \cdot \sum_{n=1}^{\infty} nx^{n-1} = x \cdot \frac{1}{(1-x)^2}$ setting $x=\frac{1}{2}$ gives 2.

(b) Express $\sum_{n=1}^{\infty} n^2 x^n$ as a *rational function* (ratio of polynomials).

Use: $\frac{d}{dx}\left(\sum_{n=1}^{\infty} nx^n\right) = \sum_{n=1}^{\infty} n^2 x^{n-1}$!

$$\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)$$

Date: 29/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(3) Find a simple formula for $\sum_{n=0}^{\infty} \frac{e^{nx}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}(e^x)^n = e^{(e^x)}$

$$e^{u} = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

Say $f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$

then $f^{(7)}(x) = \sum_{n=7}^{\infty} A_n n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(x-c)^{n-7}$

so $f^{(7)}(c) = A_7 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! \cdot A_7$ so $A_7 = \frac{f^{(7)}(c)}{7!}$

2. Taylor Series

The *Taylor series of* $f(x)$ *centered at* c is

Memorize $\rightarrow$ $$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

(4) Find the MacLaurin ($c = 0$) series of $f(x) = e^x$.

$f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$, so $A_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$,

the series is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

(5) (Final 2014) Find the Taylor series $g(x) = \log x$ centered at $\underset{c}{\cancel{a}} = 2$, as well as its radius of convergence.

$$g'(x) = \frac{1}{x}, \quad g''(x) = -\frac{1}{x^2}, \quad g'''(x) = \frac{2}{x^3}, \quad g^{(4)}(x) = -\frac{2\cdot 3}{x^4}, \quad g^{(5)}(x) = \frac{2\cdot3\cdot4}{x^5}, \ \cdots$$

$$g^{(n)}(x) \underset{\substack{\uparrow \\ n \geq 1}}{=} (-1)^{n-1} \cdot \frac{1\cdot 2\cdot 3\cdots(n-1)}{x^n} = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

$$g^{(n)}(2) = (-1)^{n-1} \cdot \frac{(n-1)!}{2^n} \quad \text{if } n \geq 1. \quad g(2) = \log 2$$

so the series is $\cancel{\Sigma}$ $\log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot(n-1)!}{2^n \cdot n!}(x-2)^n$

$$= \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n}(x-2)^n$$

For radius, $L = \lim_{n\to\infty} \left| \frac{(-1)^n}{(n+1)\cdot 2^{n+1}} \Big/ \frac{(-1)^{n-1}}{n\cdot 2^n} \right| = \lim_{n\to\infty} \frac{2^n \cdot n}{2^{n+1}(n+1)} =$

$$= \lim_{n\to\infty} \frac{1}{2}\frac{1}{1+\frac{1}{n}} = \frac{1}{2}$$

so $R = \frac{1}{L} = 2.$

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let $A_n = \begin{cases} \log 2 & n = 0 \\ \frac{(-1)^{n-1}}{n 2^n}, & \text{if } n \geq 1 \end{cases}$ then the series is $\sum_{n=0}^{\infty} A_n (x-2)^n$