28. Absolute Convergence (17/3/2017)

Goals:

(1) Review alternating series test
(2) Deciding if a series converges absolutely, conditionally, or not at all.
(3) State ratio test.

Last time: Suppose that A_n are positive, (eventually) decreasing term-by-term, and $\lim_{n\to\infty} A_n = 0$. Then:

(1) $\sum_{n=1}^{\infty}(-1)^{n-1}A_n$ converges.
(2) The error in approximating $S = \sum_{n=1}^{\infty}(-1)^{n-1}A_n$ by a partial sum $s_N = \sum_{n=1}^{N}(-1)^{n-1}A_n$ is less than the next term:

$$|s_N - S| < A_{N+1}.$$

Example: $e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$

$n!$ = number of permutations of n items

$0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 1\cdot 2\cdot 3\cdot 4 = 24$, $5! = 120$, ..

$n! = 1\cdot 2\cdot 3\cdot 4 \cdots n$, so $(n+1)! = \underbrace{1\cdot 2\cdots n}_{n!}(n+1) = n!\cdot(n+1)$

$\frac{1}{e} = \cancel{1 - 1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \frac{1}{40,320} - \ldots$

Math 101 – WORKSHEET 28
ABSOLUTE CONVERGENCE

1. More Tail Estimates

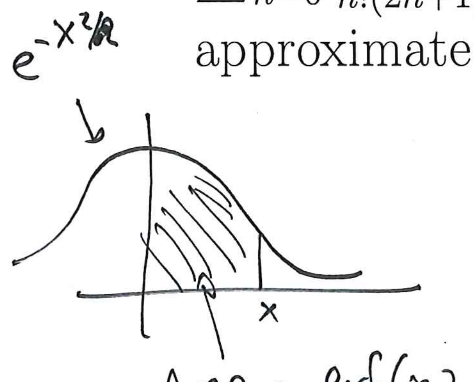
(1) It is known that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
  (a) How close is $\frac{1}{2} - \frac{1}{6} + \frac{1}{24}$ to $\frac{1}{e}$?
  (b) How many terms are needed to approximate $\frac{1}{e}$ to within $\frac{1}{1000}$?

The series $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ has alternating terms, and $\left\{\frac{1}{n!}\right\}_{n\geq 0}^{\infty}$ decrease monotonically to 0, so the AST applies. In particular,

(a) $\left|\frac{1}{e} - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)\right| \leq \frac{1}{120}$

(b) $\left|\frac{1}{e} - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)\right| \leq \frac{1}{5{,}040} < \frac{1}{1000}$. Used 7 terms

(2) The *error function* is (roughly) given by $\mathrm{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$. How many terms are needed to approximate $\mathrm{erf}(\frac{1}{10})$ to within 10^{-11}?



Absolute convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say the series converges absolutely if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges

Fact: If the series converges absolutely, it also converges

If $\sum_{n=1}^{\infty} a_n$ converges, but not absolutely, we say it converges conditionally.

Examples

Worksheet 2

Question: suppose we have $\sum_{n=1}^{\infty} \frac{100 \sin n}{n^2}$.

We'd say $\left|\frac{100 \sin n}{n^2}\right| \underset{\uparrow}{\leq} \frac{100}{n^2}$ and $\sum_{n=1}^{\infty} \frac{100}{n^2} = 100 \sum_{n=1}^{\infty} \frac{1}{n^2}$

$|\sin x| \leq 1$ converges

Note: $\frac{1{,}000}{n^2} \leq \frac{1}{n^{3/2}}$ if n is large enough

Summary: ~~Believe~~ If you believe $\sum_{n=1}^{\infty} a_n$ should converge due to rapid decay of the a_n, consider instead $\sum_{n=1}^{\infty} |a_n|$, apply

* Comparison
* limit comparison
* integral test

⋮

2. Absolute Convergence

(3) Decide if each sequence/series converges:

$\boxed{\checkmark}\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ $\quad\square\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $\quad\boxed{\checkmark}\left\{\frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ $\quad\boxed{\checkmark}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series, $p=\frac{1}{2}<1$), $\lim_{n\to\infty}\frac{(-1)^n}{\sqrt{n}}=0$ use squeeze th, converges by AST

Lesson 1: $\lim_{n\to\infty} a_n$, $\lim_{N\to\infty}\sum_{n=1}^{N} a_n$ are different

Lesson 2: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges conditionally

(4) Place checkmarks

	Converges		Diverges
	Absolutely	Conditionally	
$\sum_{n=1}^{\infty}(-1)^n$			X
$\sum_{n=1}^{\infty}\frac{1}{n^2}$	X p-series, $p=2$		
$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$	X		
$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$		AST $\Rightarrow$ con. $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ ✓	
$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$	X		
(*)$\sum_{n=1}^{\infty}\frac{\sin n}{n}$		✓	

nth element test

$0\le\left|\frac{\sin n}{n^2}\right|=\frac{|\sin n|}{n^2}\le\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

By comparison, $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}$ converges, so $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges absolutely.

(*) beyond the scope of Math 101

Ratio test

Suppose we believe a_n decay exponentially, then $\frac{a_{n+1}}{a_n}$ is about constant.

Fact: Suppose $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(1) If $r<1$, the series converges absolutely

(2) If $r>1$, the series diverges

(3) If $r=1$, the test is inconclusive

3. Ratio test

(5) Decide whether the following series converge:

(a) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(1+\frac{1}{n})}{1} = \frac{1}{2}$

$\frac{1}{2} < 1$, so by ratio test, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges

(b) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

| try: $\sum_{n=1}^{\infty} \frac{n^{100}}{2^n}$.

(c) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(d) For which values of x does $\sum_{n=0}^{\infty} nx^n$ converge?