25. Integral Test (10/3/2017)

Goals:

(1) More examples

(2) Quiz

Last time: $f(x)$ eventually positive, eventually decreasing

(for x large enough, $f(x) \geq 0$, $f'(x) \leq 0$)

Then $\sum_{n=a}^{\infty} f(n)$, $\int_a^{\infty} f(x)\,dx$ either both converge or both diverge

("two-way test")

Example: $f(x) = \frac{1}{x^p}$, $p \geq 0$. For $x>0$, $f(x)>0$, and $f'(x) = -p x^{-p-1} \leq 0$

so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} \frac{dx}{x^p}$ converge & diverge together

$$\text{converge} \Updownarrow p>1 \qquad \text{diverge} \Updownarrow p \leq 1$$

Aside: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{*}{*}\pi^4$

Math 101 – WORKSHEET 25
THE INTEGRAL TEST

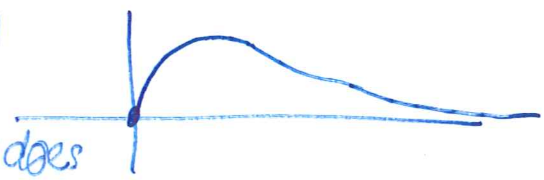
1. The integral test

(1) Decide if each series converges or diverges

(a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

Let $f(x) = \frac{x}{e^x}$. f is obviously positive for $x>0$, Also, $f'(x) = \frac{e^x - xe^x}{e^{2x}}$,

i.e. $f'(x) = \frac{1-x}{e^x}$, so $f'(x) \le 0$ if $x \ge 1$



By the integral test, $\sum_{n=1}^{\infty} f(n)$ converges if $\int_0^\infty \frac{x}{e^x}dx$ does

$$\int_0^\infty xe^{-x}dx = \lim_{T\to\infty} \int_0^T xe^{-x}dx = \lim_{T\to\infty}\left([-xe^{-x}]_0^T - \int_0^T 1(-e^{-x})\,dx\right) = \lim_{T\to\infty}\left(-Te^{-T} + \int_0^T e^{-x}dx\right)$$

by parts

$$= \lim_{T\to\infty}\left(-\frac{T}{e^T} + 1 - e^{-T}\right) = 1 - 0 - \lim_{T\to\infty}\frac{T}{e^T} = 1 - \lim_{T\to\infty}\frac{1}{e^T} = 1 \checkmark$$

l'Hôpital

(b) (Final 2014) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (your answer will depend on p!)

Let $f(x) = \frac{1}{x(\log x)^p}$. Then $f(x) > 0$ if $x > 1$, $f'(x) = -\frac{1}{x^2(\log x)^p} - \frac{p}{x^2(\log x)^{p+1}} =$

$\cancel{= -\frac{1}{x^2(\log x)^p}\left[1 + p \log x\right]}$

$$f'(x) = -\frac{\log x + p}{x^2(\log x)^{p+1}} < 0 \text{ if } \log x > -p$$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x(\log x)^p}$$

Date: 10/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.