16. More on Approximate Integrals (8/2/2017)

Goals.

(1) Writing down numerical approximations
(2) Error estimates on approximate integrals
   (a) Given f, n what is the accuracy?
   (b) Given f,what n will give error at most ϵ?

Last time. To approximate $\int_a^b f(x)\,\mathrm{d}x$ let
$\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, $\bar{x}_i = a + (i - \frac{1}{2})\Delta x$
Midpoint rule (tangent line approximation):

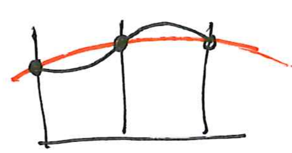
$$\Delta x\left(f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right)$$

Trapezoid rule (secant line approximation):

$$\Delta x\left(\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)$$

Simpson's rule (approximate with parabolas) – n even!

$$\frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\right)$$



2. Numerical integration

(1) (Final 2009) Let $f(x) = \sin(e^x)$. Approximate $I = \int_0^2 f(x)\,dx$ using the modpoint rule, the trapezoid rule, and Simpson's rule, with $n = 4$ in all cases. You may leave your answers in calculator-ready form.

$a = 0$, $b = 2$, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$; x_i: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

Trapezoid: $\int_0^2 f(x)\,dx \approx \frac{1}{2}\left(\frac{1}{2}\sin(e^0) + \sin(e^{\frac{1}{2}}) + \sin(e^1) + \sin(e^{3/2}) + \frac{1}{2}\sin(e^2)\right)$

Midpoint: $\approx \frac{1}{2}\left(\sin(e^{1/4}) + \sin(e^{3/4}) + \sin(e^{5/4}) + \sin(e^{7/4})\right)$

Simpson's: $\approx \frac{1}{6}\left(\sin(e^0) + 4\sin(e^{1/2}) + 2\sin(e^1) + 4\sin(e^{3/2}) + \sin(e^2)\right)$

$\frac{1}{3}\cdot\frac{1}{2}\times = \frac{1}{6}$

(2) (Final 2015) Write down the Simpson's rule approximation to $I = \int_0^2 (x-3)^5\,dx$ with $n = 6$.You may leave your answers in calculator-ready form.

$I \approx \frac{1}{9}\left((-3)^5 + 4\left(\frac{1}{3}-3\right)^5 + 2\left(\frac{2}{3}-3\right)^5 + 4(1-3)^5 + 2\left(\frac{4}{3}-3\right)^5 + 4\left(\frac{5}{3}-3\right)^5 + (2-3)^5\right)$

$\Delta x = \frac{2}{6} = \frac{1}{3}$

Question: How accurate are these approximations?

Fact: Error estimate looks like $\frac{K \cdot (b-a)^*}{M n^*}$

M = number

K = estimate on some derivative

Don't memorize: Trapezoid rule $|I - \text{approx}| \leq \frac{K(b-a)^3}{12n^2}$, $|f''(x)| \leq K$ for all x

Midpoint: $|I - \text{approx}| \leq \frac{K(b-a)^3}{24n^2}$, $|f''(x)| \leq K$.

Simpson's: $|\quad\quad| \leq \frac{K(b-a)^5}{180n^4}$, $|f^{(4)}(x)| \leq K$.

Game: Given f, find K ~~to~~ larger than specified derivative at all $a \leq x \leq b$

Math 101 – WORKSHEET 16
APPROXIMATE INTEGRATION

(1) (Final 2012) Let $I = \int_1^2 \frac{1}{x}\,\mathrm{d}x$.

(a) Write down Simpson's rule approximation for I using 4 points (call it S_4)

(b) Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \leq K$ on $[a,b]$ then the error in the approximation with n points has magnitude at most $K(b-a)^5/180n^4$.

Here, $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$, $f^{(4)}(x) = \frac{24}{x^5}$.

for $1 \leq x \leq 2$, $|f^{(4)}(x)| \leq \frac{24}{1^5} = 24$ because $f^{(4)}(x)$ is decreasing

So may take $K = 24$, so error is at most

$$\frac{24\cdot(2-1)^5}{180\cdot 4^4} = \frac{24}{180\cdot 16^2} = \frac{1}{30\cdot 16\cdot 4} = \frac{1}{30\cdot 64} \leq \frac{1}{30\cdot 60} = \frac{1}{1800}.$$

Date: 8/2/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) (Final 2015) Consider $I = \int_0^2 (x-3)^5 \, dx$.

(a) Write down the Simpson's rule approximation to I with $n = 6$. You may leave your answers in calculator-ready form.

(b) Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? Justify your answer. You may use the formulas $|E_{\mathrm{M}}| \le \frac{K(b-a)^3}{24n^2}$ and $|E_S| \le \frac{L(b-a)^5}{180n^4}$ where K is an upper bound for $|f''(x)|$ and L is an upper bound for $\left|f^{(4)}(x)\right|$.

Here $f(x)=(x-3)^5$, $f'(x)=5(x-3)^4$, $f^{(2)}(x)=20(x-3)^3$, $f^{(3)}(x)=60(x-3)^2$

$f^{(4)}(x)=120(x-3)$ x farthest from 3

for $0 \le x \le 2$, $|f^{(4)}(x)| = 120|x-3| \le 120|0-3| = 360$

$|f^{(2)}(x)| = 20|x-3|^3 \le 20\cdot 27$.

so $|E_M| \le \dfrac{20\cdot 27\cdot 2^3}{24\cdot 100^2} = \dfrac{180}{10^4} = \dfrac{18}{1000}$

simpson's ~~midpoint~~ rule gave better error bound

$|E_S| \le \dfrac{360\cdot 2^5}{180\cdot 10^4} = \dfrac{64}{10^4} = \dfrac{6.4}{1000}$

(3) (Final 2008) Let $I = \int_0^1 \cos(x^2)\,dx$. It can be shown that the fourth derivative of $\cos(x^2)$ has absolute value at most 60 on $[0,1]$. Find n such the Simpson's rule approximation to I using n points has error less than or equal to 0.001. You may use that that if $\left|f^{(4)}(t)\right| \le K$ for $a \le t \le b$ then error in using Simpson's rule to approximate $\int_a^b f(x)\,dx$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

$|\text{Error}| \le \dfrac{60(1-0)^5}{180n^4} = \dfrac{1}{3n^4}$, want n so that $\dfrac{1}{3n^4} \le \dfrac{1}{1000}$

ie $3n^4 \ge 1000$ eg. $n=10$ works

(4) Let $I = \int_4^6 \sin(\sqrt{x})\,dx$. Find n such that estimating I using the midpoint rule and n points will have an error of at most $\frac{1}{3000}$. You may use that the absolute error in estimating $\int_a^b f(x)\,dx$ using the midpoint rule and n points is at most $K(b-a)^3/24n^2$ whenever $\left|f^{(2)}(x)\right| \le K$ for $a \le x \le b$.

$f(x) = \sin(\sqrt{x})$, $f'(x) = \dfrac{1}{2\sqrt{x}}\cos(\sqrt{x})$, $f''(x) = -\dfrac{1}{4x^{3/2}}\cos(\sqrt{x}) - \dfrac{1}{4x}\sin(\sqrt{x})$

$$|f''(x)| \underset{\substack{\uparrow \\ |\sin\theta|,\,|\cos\theta| \le 1}}{\le} \frac{1}{4x^{3/2}}\cdot 1 + \frac{1}{4x}\,1 \underset{\substack{\uparrow \\ 4 \le x \le 6}}{\le} \frac{1}{4\cdot 4^{3/2}} + \frac{1}{16} = \frac{1}{16} + \frac{1}{32} = \frac{3}{32} \le \frac{1}{10}$$