4. THE FUNDAMENTAL THEOREM OF CALCULUS (11/1/2017)

Goals.

(1) Review of definite integral
(2) State and justify FTC I
(3) Differentiate integrals
(4) Deduce FTC II
(5) Compute integral using anti-derivatives.

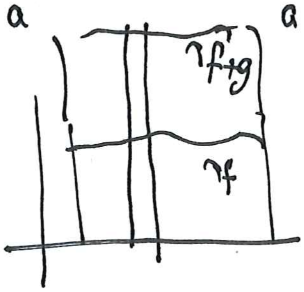
Last Time.

- The Integral as a limit of Riemann sums
- Evaluating Integrals by definition
- Evaluating Integrals as areas

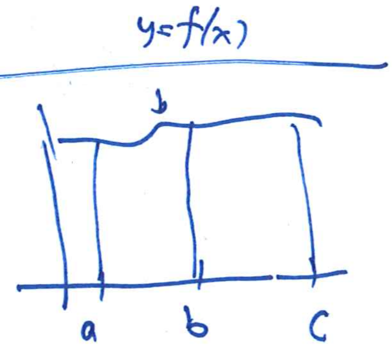
Q: When are office hrs? A: Tue 10-11, Wed 11-12:30

(1) Properties of the integral

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx \qquad \Big| \qquad \int_a^b (\alpha f)\,dx = \alpha \int_a^b f\,dx$$



$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$$



$$\int_b^a f\,dx = -\int_a^b f(x)$$

check consistency: $\int_a^b f\,dx + \int_b^a f\,dx = \int_a^a f\,dx = 0$

Math 101 – WORKSHEET 4
THE FUNDAMENTAL THEOREM OF CALCULUS

(1) (Differentiating integrals) Evaluate

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{t^2}\,\mathrm{d}t$ $= e^{x^2}$

$f(t) = e^{t^2}$ $\quad \frac{d}{dt}\int^x f(t)dt = f(x) = e^{x^2}$

what about $\int_1^x e^{t^2}dt$? $\quad \int_0^x e^{t^2}dt - \int_1^x e^{t^2}dt = \int_0^1 e^{t^2}dt$ = constant.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 e^{t^2}\,\mathrm{d}t$ $= \frac{d}{dx}\left(-\int_1^x e^{t^2}dt\right) = -e^{x^2}$

common error: $\frac{d}{dx}\int_0^x e^{t^2}dt \neq e^{t^2}\cdot 2t$

(c) (Final 2009) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{e^x}\sqrt{\cos t}\,\mathrm{d}t$ $= \sqrt{\cos(e^x)}\cdot \underset{\frac{d(e^x)}{dx}}{e^x} - \sqrt{\cos(x^2)}\cdot \underset{\frac{d(x^2)}{dx}}{2x}$

$\int_{x^2}^{e^x} = \int_{x^2}^{1} + \int_1^{e^x} = \int_1^{e^x} - \int_1^{x^2}$

Alternative: let $F(u) = \int^u \sqrt{\cos t}\,dt$. Find $\frac{d}{dx}\left(F(e^x) - F(x^2)\right)$

(d) (Final 2014) Let $f(x) = \int_1^x 100(t^2-3t+2)e^{-t^2}\,\mathrm{d}t$.
Find the interval(s) on which f is increasing.

$\frac{df}{dx} = 100(x^2-3x+2)e^{-x^2} = 100e^{-x^2}(x-1)(x-2)$

Also come from props of Σ :

$$\sum_{i=1}^{n}(i^2+3i) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i$$

The Fundamental Thm of Calculus

(bottom line: can use derivatives to calculate integrals)

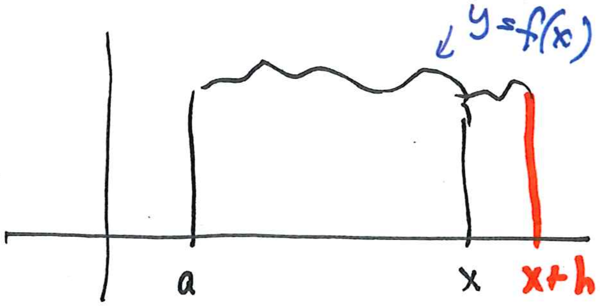
Idea: (1) shift bounds of integration get a function

$$F(x) = \int_a^x f(t)\,dt$$

different vars!

(2) Study F using diff calc.

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$



$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

if h small
f cts $\rightarrow$ $\approx h \cdot f(x)$

so $\dfrac{F(x+h)-F(x)}{h} \approx \dfrac{hf(x)}{h} = f(x)$

<u>Thm</u>: If f is cts, $F(x) = \int_a^x f(t)\,dt$ is <u>diff</u> and $F'(x) = f(x)$

Discovery: if $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$

So $F(x)$ is an anti-derivative of f.

If G is another anti-derivative, then $G(x)=F(x)+C$

because $(F-G)'=f-f=0$

Suppose G is some anti-derivative Then $F(x)=G(x)+C$

But $F(a)=\int_a^a f\,dt=0$ so $G(a)+C=0$ so $C=-G(a)$,

$F(x)=G(x)-G(a)$, ie.

$$\boxed{\int_a^b f(t)\,dt=G(b)-G(a)}$$

G any anti-derivative.

(2) Evaluate using anti-derivatives

(a) (Final 2012) $\int_1^2 \frac{x^2+2}{x^2}\,dx = \int_1^2 \left(1+\frac{2}{x^2}\right)dx$

side calc: $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$= x - \frac{2}{x}\Big|_1^2 = \left(2-\frac{2}{2}\right) - \left(1-\frac{2}{1}\right)$

$= 1 - (-1) = 2$

massaging

$= \left[x - \frac{2}{x}\right]_{x=1}^{x=2}$

different notation

(b) (Final 2007) $\int_{-1}^{0}(2x - e^x)\,dx =$

(c) $\int_3^{10}\left(x^{5/2}+e^{2x}\right)dx =$