Math 539: Problem Set 0 (due 18/1/2016)

Policy for Problem Sets: adopt a reasonable workload for your situation; you should do some non-trivial problems in any case.

In this problem set: We will repeatedly rely on this results of this problem set; there is no number theory here yet. If you only do some of the problems, I recommend starting with 1,3 and parts of of 5 and 6. Problem 7 is intended to review ideas from Math 437/537.

Real analysis

1. (Asymtotic notation) Let \( f, g \) be defined for \( x \) large enough. We write \( f \ll g \) and \( f = O(g) \) if there is \( C > 0 \) such that \( |f(x)| \leq Cg(x) \) for all large enough \( x \).

(a) Let \( f, g \) be functions such that \( f(x), g(x) > 2 \) for \( x \) large enough. Show that \( f \ll g \) implies \( \log f \ll \log g \). Give a counterexample under the weaker hypothesis \( f(x), g(x) > 1 \).

(b) For all \( A > 0, 0 < b < 1 \) and \( \epsilon > 0 \) show that for \( x \geq 1 \),

\[
\log^A x \ll \exp\left( \log^b x \right) \ll x^\epsilon.
\]

2. Set \( \log_1 x = \log x \) and for \( x \) large enough, \( \log_{k+1} x = \log (\log_k x) \). Fix \( \epsilon > 0 \).

(PRAC) Find the interval of definition of \( \log_k x \). For the rest of the problem we suppose that \( \log_k x \) is defined at \( N \).

(a) Show that \( \sum_{n=N}^\infty \frac{1}{n\log n \cdots \log_{k-1} n \log_k n^{1+\epsilon}} \) converges.

(b) Show that \( \sum_{n=N}^\infty \frac{1}{n\log n \cdots \log_{k-1} n \log_k n^{1+\epsilon}} \) diverges.

3. (Stirling’s formula)

(a) Show that \( \int_{k-1/2}^{k+1/2} \log t \, dt - \log k = O\left( \frac{1}{k^2} \right) \).

(b) Show that there is a constant \( C \) such that

\[
\log(n!) = \sum_{k=1}^n \log k = \left( n + \frac{1}{2} \right) \log n - n + C + O\left( \frac{1}{n} \right).
\]

RMK \( C = \frac{1}{2} \log(2\pi) \) (see problem 6(f) below) but this is rarely relevant.

4. Let \( \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subset \mathbb{C} \) be sequences with partial sums \( A_n = \sum_{k=1}^n a_k, B_n = \sum_{k=1}^n b_k \).

(a) (Abel summation formula) \( \sum_{n=1}^N a_nb_n = A_N b_N - \sum_{n=1}^{N-1} A_n (b_{n+1} - b_n) \)

– (Summation by parts formula) Show that \( \sum_{n=1}^N a_nb_n = A_N b_N - \sum_{n=1}^{N-1} A_n b_{n+1} \).

(b) (Dirichlet’s test) Suppose that \( \{A_n\}_{n=1}^\infty \) are uniformly bounded and that \( b_n \in \mathbb{R}_{>0} \) decrease monotonically to zero. Show that \( \sum_{n=1}^\infty a_nb_n \) converges.

(c) Let \( \chi_3(n) = \begin{cases} 
1 & n \equiv \pm 1 \ (3) \\
0 & 3|n 
\end{cases} \).

Show that \( \text{Dirichlet’s L-series} \ L(s, \chi_3) = \sum_{n=1}^\infty \chi_3(n)n^{-s} \) converges for \( s \) real and positive.


Complex analysis: the Gamma function

DEFINITION. The *Mellin transform* of a function $\phi$ on $(0, \infty)$ is given by $\mathcal{M}\phi(s) = \int_0^\infty \phi(x)x^s \frac{dx}{x}$ whenever the integral converges absolutely.

5. Let $\phi$ be a bounded function on $(0, \infty)$ [measurable so the integrals make sense]
   (a) Suppose that for some $\alpha > 0$ we have $\phi(x) = O(x^{-\alpha})$ as $x \to \infty$ (see problem 1 for this notation). Show that the $\mathcal{M}\phi$ defines a holomorphic function in the strip $0 < \Re(s) < \alpha$. For the rest of the problem assume that $\phi(x) = O(x^{-\alpha})$ holds for all $\alpha > 0$.
   (b) Suppose that $\phi$ is smooth in some interval $[0, b]$ (that is, there $b > 0$ and is a function $\psi \in C^\infty([0, b])$ such that $\psi(x) = \phi(x)$ with $0 < x \leq b$). Show that $\hat{\phi}(s)$ extends to a meromorphic function in $\Re(s) < \alpha$, with at most simple poles at $-m, m \in \mathbb{Z}_{\geq 0}$ where the residues are $\frac{\phi(m)(0)}{m!}$ (in particular, if this derivative vanishes there is no pole).
   (c) Extend the result of (b) to $\phi$ such that $\phi(x) - \sum_{n=1}^\infty \frac{\alpha}{x^n}$ is smooth in an interval $[0, b]$.
   (d) Let $\Gamma(s) = \int_0^\infty e^{-xt}x^{s-1}dt$. Show that $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $\mathbb{Z}_{\leq 0}$ where the residue at $-m$ is $\frac{(-1)^m}{m!}$.

6. (The Gamma function) Let $\Gamma(s) = \int_0^\infty e^{-xt}x^{s-1}dt$, defined initially for $\Re(s) > 0$. See supplementary problem B for a proof that this extends to a meromorphic function in $\mathbb{C}$ and a determination of the location and residues at the poles (all poles are simple).
   FACT A standard integration by parts shows that $s\Gamma(s) = \Gamma(s+1)$ and hence $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\geq 1}$.
   (a) Let $Q_N(s) = \int_0^N (1 - \frac{x}{N})^N x^s \frac{dx}{x}$. Show that $Q_N(s) = \frac{N^1}{s(s+1)\cdots(s+N)}N^s$. Show that $0 \leq \left(1 - \frac{x}{N}\right)^N \leq e^{-x}$ holds for $0 \leq x \leq N$, and conclude that $\lim_{N \to \infty} \frac{N^1}{s(s+1)\cdots(s+N)}N^s = \Gamma(s)$ for on $\Re(s) > 0$ (for a quantitative argument show instead $0 \leq e^{-x} - \left(1 - \frac{x}{N}\right)^N \leq \frac{x^2}{N}e^{-x}$).
   (b) Define $f(s) = se^{\gamma s} \prod_{n=1}^\infty (1 + \frac{s}{n}) e^{-s/n}$ where $\gamma = \lim_{n \to \infty} \left(\sum_{n=1}^\infty \frac{1}{n^2} - \log n\right)$ is Euler’s constant. Show that the product converges locally uniformly absolutely and hence defines an entire function in the complex plane, with zeros at $\mathbb{Z}_{\leq 0}$ . Show that $f(s+1) = \frac{1}{s}f(s)$.
   (c) Let $P_N(s) = se^{\gamma s} \prod_{n=1}^\infty (1 + \frac{s}{n}) e^{-s/n}$. Show that for $\alpha \in (0, \infty), \lim_{N \to \infty} Q_N(\alpha)P_N(\alpha) = 1$ and conclude (without using problem 5!) that $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $\mathbb{Z}_{\leq 0}$, that $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and that it has the Weierstraß product representation
   $$
   \Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}.
   $$
   (d) Let $F(s) = \frac{\Gamma(s)}{\Gamma(1+s)}$ be the Digamma function. Using the Euler–Maclaurin summation formula
   $$
   \sum_{n=0}^N f(n) = \int_0^N f(x) dx + \frac{1}{2} (f(0) + f(N)) + \frac{1}{12} (f'(0) - f'(N)) + R,
   $$
   with $|R| \leq \frac{1}{12} \int_0^N |f''(x)| dx$, show that if $-\pi + \delta \leq \arg(s) \leq \pi + \delta$ and $s$ is non-zero then
   $$
   F(s) = \log s - \frac{1}{2s} + O_\delta \left(|s|^{-2}\right).
   $$
Integrating on an appropriate contour, obtain Stirling’s Approximation: there is a constant \( c \) such that if \( \arg(s) \) is as above then

\[
\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + c + O(\delta) \left( \frac{1}{|s|} \right).
\]

RMK Compare with the result of 3(b)

(e) Show Euler’s reflection formula

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.
\]

Conclude that \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \) and hence that \( \int_{-\infty}^{+\infty} e^{-\alpha x^2} \, dx = \sqrt{\frac{\pi}{\alpha}} \).

(f) Setting \( s = \frac{1}{2} + it \) in the reflection formula and letting \( t \to \infty \), show that \( c = \frac{1}{2} \log(2\pi) \) in Stirling’s Approximation.

(g) Show Legendre’s duplication formula

\[
\Gamma \left( \frac{s}{2} \right)\Gamma \left( \frac{s+1}{2} \right) = \sqrt{\pi} 2^{1-s} \Gamma(s).
\]

Review of arithmetic functions

REMARK We won’t do any serious abstract algebra in this course, but I will use basic terminology like “commutative ring”. For definitions Wikipedia is your friend.

7. Most of the stuff below should be familiar from Math 437/537

DEF (Dirichlet convolution) \( (f * g)(n) = \sum_{ab=n} f(a)g(b) \).

(a) The set of arithmetic functions with pointwise addition and Dirichlet convolution forms a commutative ring. The identity element is the function \( \delta(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases} \).

(b) \( f \) is invertible in this ring iff \( f(1) \) is invertible in \( \mathbb{C} \).

(c) If \( f, g \) are multiplicative so is \( f * g \).

DEF \( I(n) = 1, N(n) = n, \phi(n) = \left| \left( \mathbb{Z}/n\mathbb{Z} \right) \times \right|, \mu(n) = (-1)^r \) if \( n \) is a product of \( r \geq 0 \) distinct primes, \( \mu(n) = 0 \) otherwise (i.e. if \( n \) is divisible by some \( p^2 \)).

(d) (“Möbius inversion”) Show that \( I * \mu = \delta \) by explicitly evaluating the convolution at \( n = p^m \) and using (c).

(e) Show that \( \phi * I = N \): (i) by explicitly evaluating the convolution at \( n = p^m \) and using (c); (ii) by a combinatorial argument.

DEF A derivation in the ring \( R \) is a function \( D: R \to R \) such that for all \( f, g \in R \) one has \( D(fg) = Df \cdot g + f \cdot Dg \) (example: \( D = \frac{d}{dx} \) acting on smooth functions).

(f) Show that pointwise multiplication by an arithmetic function \( L(n) \) is a derivation in the ring of arithmetic functions iff \( L(n) \) is completely additive: \( L(de) = L(d) + (e) \) for all \( d, e \geq 1 \). In particular, this applies to \( L(n) = \log n \).
Supplementary problems are not for submission.

**Definition.** The *ring of Dirichlet polynomials* (let’s denote it $\mathcal{D}_f$) consists of all formal expressions of the form $D(s) = \sum_{n \leq x} a_n n^{-s}$ where $a_n \in \mathbb{C}$ (modulo the obvious equivalence relation). Call $a_1$ the *constant coefficient*. Multiplication is the bilinear map induced from $n^{-s} \times m^{-s} = (nm)^{-s}$.

**A. (Basics)**

(a) Show that this is a ring, and that the map $D(s) \mapsto a_1$ is a ring homomorphism $\mathcal{Q}: \mathcal{D}_f \to \mathbb{C}$. Write $m$ for its kernel, the maximal ideal.

(b) Conversely, show that every homomorphism $\mathcal{D}_f \to \mathbb{C}$ is of the form $\sigma \circ \mathcal{Q}$ for some $q \in \text{Aut}(\mathbb{C})$.

(c) Let $\mathcal{V} : [0, \infty]$ be given by $\mathcal{V}(\sum_n a_n n^{-s}) = N$ if $a_N \neq 0$ but $a_n = 0$ for $n < N$ (set $\mathcal{V}(0) = \infty$). Show that $\mathcal{V}(D_1 + D_2) \geq \min \{\mathcal{V}(D_1), \mathcal{V}(D_2)\}$ and conclude that $d(D_1, D_2) = \exp \{\mathcal{V}(D_1 - D_2)\}$ is a metric on $\mathcal{D}_f$ (in fact, an ultrametric).

(d) Show that the ring $\mathcal{D}$ of Dirichlet series is exactly the completion of $\mathcal{D}_f$ with respect to the metric.

(e) Show that for any arithmetic function $f$, the series $\sum_{n \geq 1} f(n) n^{-s}$ (thought of as a sum of the individual Dirichlet polynomials $f(n) n^{-s}$) converges in $\mathcal{D}$ to formal series $D_f(s) = \sum_{n \geq 1} f(n) n^{-s}$.

(f) (Calculus student’s dream) Show that (for $D_i \in cD$) a series $\sum_i D_i$ converges iff the terms converge to zero (i.e. iff $\mathcal{V}(D_i) \to \infty$).

(f) Show that the product $\prod_i (1 + D_i)$ converges and diverges under the same hypothesis.

**B. (exp and log)**

(a) Let $F(T) \in T^\mathbb{C}[[T]]$ be a formal power series with no constant coefficient, say $F(T) = \sum_{k=1}^\infty a_k T^k$, and let $D \in m$ be a formal Dirichlet series with no constant coefficient. Show that $F(D) = \sum_{k=1}^\infty a_k D^k$ converges in our topology to an element of $frakm$, so that $F : m \to m$ is continuous.

(b) Show that same for a two-variable power series with no constant coefficient, $G(T, S) \in (T + S)^\mathbb{C}[[[T, S]]].$

(c) Conclude that $\log(1 + D)$, $\exp(D)$ exist for $D \in m$ and satisfy $\log((1 + D_1)(1 + D_2)) = \log(1 + D_1) + \log(1 + D_2)$ and $\exp(D_1 + D_2) = \exp(D_1) \exp(D_2)$.

(d) Show that the construction above respects composition of formal power series with no constant coefficient, and conclude that $\exp \log D = D$ and that $\log \exp D = D$.

(e) Extend exp to all of $\mathcal{D}$ using the topology of pointwise convergence of the coefficients.

(f) The *informal derivative* of $D(s) = \sum_{n \geq 1} f(n) n^{-s} \in \mathcal{D}$ is the series $D'(s) = \sum_{n \geq 1} (f(n) \log(n)) n^{-s}$.

In 7(f) you obtained the Leibnitz identity $(D_1D_2)' = D_1'D_2 + D_1D_2'$. Show that $(\log D)' = \frac{D'}{D}$ and that $(\exp D)' = (\exp D)D'$.

**C. (Euler products)**

(a) For each prime $p$ let $D_p$ be a formal Dirichlet series supported on the powers of $p$, with constant coefficient 1. Show that $\prod_p D_p$ converges in $cD$. Series obtained this way are said to have an *Euler product*.

(b) Show that every series has at most one representation as an Euler product, and that if $D_1, D_2$ have an Euler product then so does $D_1D_2$. 

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