3/12/2015

A: half scored in [27, 40] 160

Last time: Solvable gps:

\[ G \text{ solvable } \iff \text{normal series } \exists \{G_0, G_1, \ldots, G_n = G\} \]

with \( G_n/G_i \text{ abelian.} \)

\[ \text{Saw } G \text{ solvable } \implies H \leq G, \ G/N \text{ solvable} \]

(2) No G, if \( N, G/N \text{ solvable so is } G \).

Example: \( B_n \equiv \begin{pmatrix} 1 & \ast & \ast \\ 0 & 1 & \ast \\ 0 & 0 & 1 \end{pmatrix} \subseteq GL_n(F) \) (upper-triangular matrices)

HW: \( U_n \triangleleft B_n, \ B_n/U_n = (F^*) \)

Solubility top-down: relabel series \( G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\} \)

want \( G_i/G_{i-1} \) to be abelian, want for any \( x, y \in G \) that \( [x, y] = e \)

\( x = x_{G_i}, \ y = y_{G_i} \), images of \( x, y \) in \( G/G_i \).

\( \iff \) want \( [x, y] \in G_i \).

\( \Rightarrow G/G_i \text{ abelian iff } G_i [x, y] \subseteq [x, y] \forall x, y \in G \).

\( \iff G_i \subseteq [x, y] \cdot [x, y] = [G_i, G_i] \subseteq G_i \).

HW: if \( G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\} \) \( \text{"derived subgp."} \)

\( G/G_i \text{ abelian then } G_i 
\subseteq G^{(i)} : G^{(0)} = G, \ G^{(i+1)} = (G^{(i)})' \)

clearly, \( G^{(i)}/G^{(i+1)} \) commutative: killed all commutators

Conclusion \( G \text{ solvable iff } G^{(n)} = \{e\} \) for some \( n \).
$G^{(1)}$ generated by $[x, y]$

$G^{(2)} = [ [x, y], [x', y'] ]$

checks $(B_n)' = U_n$

corollary: $S^n(B_n) = (G_n(\mathbb{F}))' \circ B_n(B_n)' = U_n$

also: $det(x'y'x^{-1}y^{-1}) = 1$, so $[x, y] \in Ker(det) = SL_n(\mathbb{F}) = \{ g \in GL_n(\mathbb{F}) \mid det g = 1 \}$

$G'$ normal; $\phi \in Aut(G)$. Then $\phi([x, y]) = [\phi(x), \phi(y)]$

then $\phi$ preserves the commutators.

so fixed the subgroup they generate; $\phi(G') = G'$

What is the normal subgroup generated by $U_n(\mathbb{F})$?

Over $\mathbb{R}$, $U_n(\mathbb{R}) = upper \ tri$angular, $\overline{U_n} = lower - tri$angular

$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

jointly generate $SL_n(\mathbb{R})$ (Gaussian elimination)
Group theory in topology

Topology: study of properties unchanged by deformation.

Basic question: given \( S, \gamma \), want to know: are they the same?

Let \( \pi \) be the fundamental group

\((\check{\Delta}, \pi)\) is a (topological space) + pt \(*\)

Consider the set of based loops \( C((S^1, \ast), (\check{\Delta}, \pi)) \)

(cts maps \( \gamma : [0, 1] \to \check{\Delta} \), s.t. \( \gamma(0) = \ast, \gamma(1) = \ast \))

natural operation: concatenation: if \( \gamma_1, \gamma_2 \) loops define

\( \gamma_1 \cdot \gamma_2 \) do \( \theta \), then \( \gamma_2 \).

Associative: \((\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)\)

Identity: constant loop \( \ast \) e(\(t\)) = \( \ast \) for all \( t \).

Inverse: \( \gamma(\(t\)) = \gamma(1-t) \) (reverse direction)

declare \( \gamma_1 \sim \gamma_2 \) if can deform \( \gamma_1 \) to \( \gamma_2 \) (endpoints fixed)

deform \( \gamma \) to \( \epsilon \), at time \( 0 \)

at time \( \epsilon \)

at time \( 1-\epsilon \)

at time \( 1 \)
Combining topics

G, gp, S generating set. (Assume symmetric: if $s \in S$, $s^{-1} \in S$ too)

Definition: Cayley graph $\text{Cay}(G; S)$ is the graph with vertex set $G$

$$E = \{(g, gs) \mid g \in G, s \in S\}$$

(Graph: pair $P = (V, E)$ V vertices, E: edge connect vertices

Cayley $\text{Cay}(\mathbb{Z}, \{2, 1\})$ edges: $(n, n \pm 1)$

\[ \begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array} \]

\[ \text{Cay}(\mathbb{Z}/n\mathbb{Z}, \{2, 1\}) = \text{in-cycle} \]

If $(g, gs) \in E$ then $(gs, g^{-1}s^{-1}) = (gs, g) \in E$ too

$G$ acts on $\text{Cay}(G; S)$: translation $g \cdot x = gx$

maps edge $(x, xs)$ to edge $(gx, g(xs))$

Example: (up to a congruence condition) $\text{SL}_2(\mathbb{F}_q)$, $q$ prime,

has a generating set of size $p+1$ with $\text{Cay}(\text{SL}_2(\mathbb{F}_q); S)$

extremely well-connected:

This network has $q^3$ vertices, each with only $p+1$ neighbours
(think $p$ fixed, $q \to \infty$) but if cut it into two pieces A, B

many connections across the cut.
Check 1) is an equiv rel
2) if \( x_1 \sim x_2 \) the \( y_1 \cdot y_2 = y_1' \cdot y_2' \)

follows: still well-defined on \( C(S^1, X, x_0) \), \([e] \) still identity

now \( [x \circ f] \cdot [y \circ g] = [x \circ (f \cdot g)] = [e] \)

so we got a group! Call it \( \pi_1(X, x) \).

Examples: \( \pi_1(S^1) = \mathbb{Z} \) (only loop on \( S^1 \), need to know winding number)
\( \pi_1(S^2) = \mathbb{Z} \)
\( \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \)

suppose \( x, y \) points in \( X \) connected by path \( p \)

\( \circlearrowleft \) give loop \( \vec{p} \) based at \( y \)

the loop \( \vec{p} : x \to \vec{p} \) is also based at \( x \)

converse identification uses \( \vec{p} \), gives these an inverse modulo deformation:

\( \pi_1(X, x) = \pi_1(X, y) \)
\( \pi_1(X, x) = \left( C(S^1) / \sim \right) \times \mathbb{Z} \)
\( \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) \)
\[ \mathcal{H} = \{ z = x + iy \mid y > 0 \} \]

g \in SL_2(\mathbb{R}) : g = (a \ b \\
    c \ d) \quad \text{define} \quad g \cdot z = \frac{az + b}{cw + d}

This is a group action on \( \mathcal{H} \).

(preserves distance) action transitive, \( \text{Stab}_{SL_2(\mathbb{R})}(i) = SO(2) \)

In particular, \( SL_2(\mathbb{Z}) \) acts.

\[ \begin{align*}
    \left( \begin{array}{cc} 1 & 1 \\
    0 & 1 \end{array} \right) & \quad \left( \begin{array}{cc} 1 & 0 \\
    0 & 1 \end{array} \right) \quad \left( \begin{array}{cc} -1 & 0 \\
    0 & -1 \end{array} \right) \\
    \frac{1 - \sqrt{3}}{2} & \quad 0 & \quad \frac{1 + \sqrt{3}}{2}
\end{align*} \]

Example: count solutions to \( x^2 + y^2 + z^2 + w^2 = N \) \((x, y, z, w) \in \mathbb{Z}^4)\)

using holomorphic forms on \( SL_2(\mathbb{Z}) \) \( \mathcal{H} \).