Two from PS10:

1) (c): A abelian, \( A/\text{Atoms} \) is torsion-free

\[ \text{Pf: } q : A \to A/\text{Atoms} \text{ is the quotient map} \]

suppose \( \bar{a} \in (A/\text{Atoms})_{\text{tors}} \), say \( \bar{a} = q(a) \)

By assumption, for some \( k \to 0 \), \( \bar{a}^k = e \), i.e. \( q(a)^k = e \), i.e. \( q(a) = e \), so \( a^k \in \text{Ker}(q) \), i.e. \( a^k \in \text{Atoms} \)

This means \( \exists l \) s.t. \( (a^l)^k = e \), so \( a = e \), \( a \in \text{Atoms} \), and \( \bar{a} = q(a) = e \).

Observe: we showed: if \( G \) any gp, \( N \triangleleft G \), \( N \subset G_{\text{tors}} \) and if \( g \in G \) is torsion mod \( N \), then \( g \in G_{\text{tors}} \)

\[ gN \in (G/N)_{\text{tors}} \]

4) (b): Say \( G/Z(G) \) abelian. Show \( G_{\text{tors}} \) is a subgroup

\[ \text{Pf: let } xy \in G_{\text{tors}}. \text{ Need to show } xy \in G_{\text{tors}} \]

Saw: \( [x,y] \in Z(G)_{\text{tors}} \), if \( [x,y] = e \) then \( xy \in G_{\text{tors}} \)

Consider images \( \bar{x}, \bar{y} \) of \( x, y \) in \( G/Z(G)_{\text{tors}} \).

\( Z(G)_{\text{tors}} \) is a subgroup of \( Z(G) \) (\( Z(G) \) is abelian)

is normal in \( G \) because for \( g \in G, x \in Z(G)_{\text{tors}} \), \( g^{-1}xg = x \)

\( x, y \) torsion in \( G/Z(G)_{\text{tors}} \) (in general: if \( x = e \) then \( f(x) = e \) for any hom \( f : G \to H \))

Also, \( x, y \) commute:

\[ [x,y] = [q(x), q(y)] = q(x)q(y)q(x)^{-1}q(y)^{-1} = q(xyx^{-1}y^{-1}) = q([x,y]) = e \]

where \( q : G \to G/Z(G)_{\text{tors}} \) is quot. map
so \( xy = g(xy) \) is torsion (if \( x \neq e, y \neq e \), \( (xy)^k = e \))

By observation above, \( xy \in G_{\text{tors}} \) too.

Suppose \( G/\tau(G) \) is two step-nilpotent.
(say "\( G \) is three-step nilpotent"). Again \( G_{\text{tors}} \) is a subgroup.

Let \( x, y \in G_{\text{tors}} \). Consider images of \( x, y \) in \( G/\tau(G) \). These are torsion elements there, by \( (G/\tau(G))_{\text{tors}} \) is a subgroup, so \( (xy)^t \) is torsion, i.e. \( (xy) \in \tau(G) \) for some \( k \).

Not done. don't know \( (xy)^k \in \tau(G)_{\text{tors}} \).

Def: \( G \) is \( k \)-step nilpotent if \( G = Z(G) \) but \( G/\tau(G) \) is.

Def: \( G \) is \((k+1)\)-step nilpotent if \( G \) is not \( k \)-step nilpotent. \( G/\tau(G) \) is.

Example: Finite \( p \)-groups are nilpotent.

Pri: By induction on order: if \( G \) finite \( p \)-gp. \( Z(G) \neq \{1\} \), show: \( G \) nilp, \( G/\tau(G) \) nilp.

So \( G/\tau(G) \) is smaller, by induction nilpotent.

Fact: Finite gp \( G \) is nilpotent iff \( G = \prod_{i=1}^{n} P_i \) (Part of Sylow Subgps.)

Further Study \( \tau(G) \), \( G/\tau(G) \), put together.

Suppose \( G \) is \( k \)-step nilpotent. Let \( \tau_i(G) = \{ z \in G \mid z^n = 1 \} \)
define \( \tau_{i+1}(G) \) to be the subgroup \( \tau_{i+1}(G) = \tau(G)/\tau_i(G) \).

Correspondence: \( \{ \text{subgps of } G \} \) containing \( \tau(G) \)
\( \leftrightarrow \{ \text{subgps of } G/\tau(G) \} \).

Notes: \( \tau_2(G) = \{ z \in G \mid z^2 \in \text{central "mod centre"} \} \), \( \tau_1(G) = \{ z \in G \mid \forall g \in G : [z, g] \in \tau(G) \} \).

Thus, \( \tau_i(G) \) is abelian by \( \tau_{i+1}(G)/\tau_i(G) = Z(G/\tau_i(G)) \).
Different view of nilpotence: For any $G$, can define $\delta_0, \delta_1, \delta_2, \ldots$.

$G$ is nilpotent if $\delta_k(G) = G$ for some $k$.

"lower central series".

Notes: $\delta_i(G)$ normal in $G$, $\delta_i(G)/\delta_{i+1}(G)$ commutative.

General nonlinear:

Example. Reminder (linear algebra) $T \in \text{End}(V)$ is nilpotent when $T^k = 0$ for some $k$.

Example:

$U_n = \{ g \in \text{GL}(n, \mathbb{R}) \mid \text{g upper-triangular} \}$ if $g \in U_n, (g-I)$

$U_2 = \{ \binom{1}{0} \}, \ U_3 = \{ \binom{1}{0} \}$

$\mathcal{Z}(U_3) = \{ \binom{0}{0} \} \ U_3/\mathcal{Z}(U_3) = \{ \binom{1}{0} \}$

$\mathbb{F}_2 = \{ 0, 1 \}$

$g = \binom{1}{0}$

$\binom{1}{0} \delta = \binom{1}{0}$

$T(U_3) = \mathcal{Z}(U_3)$

$T(U_4) = \mathcal{Z}(U_4)$

$\mathcal{Z}(U_4) = U_4$

or $g \in U_n$, $\log(g) = \log(I + (g-I)) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} (g-I)^i$.
Def: \( G \) is agp. Call a chain of subgroups
\[ \{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_k = G \]
a normal series if \( G_i \triangleleft G_{i+1} \) for each \( i \).
(don't need \( G_i \triangleleft G \)).

Always have \( \{e\} \triangleleft G \).

Idea: Understand \( G \) from quotients \( G_{i+1}/G_i \).

Def: Say \( G \) is solvable if \( G \) has a normal series with
\( G_i = G_{i+1}/G_i \) abelian for all \( i \).

Example: nilpotent gps. Also \( B_n = \{ g \in GL_n \mid g \text{ upper triangular} \} \).

Say \( G \) is solvable of deg \( d \) if has normal series with \( d \) terms
& abelian quotients.

Thm (Galois): Let \( f \in \mathbb{Q}[x] \) be a polynomial, let \( \Sigma \subset \mathbb{C} \) be
the field generated by roots of \( f \). ("splitting field of \( f \")
\[ \text{let } \text{Gal}(f) = \text{Aut}(\Sigma) = \{ \phi : \Sigma \to \Sigma | \phi \text{ bijective} \} \]
\[ \phi(x+y) = \phi(x) + \phi(y) \]
\[ \phi(xy) = \phi(x) \phi(y) \]

Then, (i) \# Gal(f) finite.
(ii) Roots of \( f \) can be expressed using radicals.

(\( \text{iff } \text{Gal}(f) \) is solvable.

Example: \( f(x) = ax^2 + bx + c \), roots \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
(if \( \sqrt{b^2 - 4ac} \in \mathbb{Q} \), \( \text{Gal}(f) = \mathbb{Z}/2 \))
(if \( \sqrt{b^2 - 4ac} \notin \mathbb{Q} \), \( \text{Gal}(f) = C_2 \))

\( \text{bely } \text{Gal}(f) \) commutative \( \Rightarrow f \) is solvable by radicals.