Problem: Let $G$ be a finite group, $M \triangleleft G$ of index $p$, where $p$ is the smallest prime dividing $|G|$. Then $M \triangleleft G$.

Solution: Let $G$ act on $\Xi=G/M$ by translation.

Get homomorphism $f: G \to S_\Xi = S_p$ $(\#\Xi = [G:M]=p)$

Note: $(\#G, \#S_p) = p$, try to make a subgroup $S_p$ of order dividing $|G|$.

Let $H = f(G) = \text{Im}(f) < S_p$.

(a) $\#H | \#G$ because by 1st isom thm, $H = G/\ker(f)$ so $\#H = [G: \ker(f)] | \#G$

(b) $\#H \leq p! = \#S_p$ by Lagrange

but $p! = (p-1)! \cdot p$, and $(p-1)!$ is prime to $\#G$.

so $\#H | p$, so $\#H$ is either 1 or $p$.

But $G$ acts transitively on $G/M$, so $H$ acts transitively, so $\#H = p$.

For case if $g \in G$, $g \not\in M$ then $gM \neq M$ so $f(g) \neq \text{id}_\Xi$ so $H \neq \text{id}_\Xi$.

By 1st isom thm, if $\text{Im}(f)$ has order $p$, $[G: \ker(f)] = p$.

But $[G:M] = p$. Now if $g \not\in M$ then $g \not\in \ker f$, so

$\ker(f) \subseteq M$.

so $[M: \ker(f)] = \frac{[G: \ker(f)]}{[G:M]} = 1$ and $M = \ker(f)$ is normal.
Classification of groups of order 12

Recap: let $G$ be a finite group of order $m = p^r m'$, $p$ prime, $p 
mid m'$, $r \geq 1$

Then:
1. $G$ has subgroups of order $p^r$.
2. They are all conjugate, their number $n_p(G) | m$.
3. $n_p(G) = 1$ (p)

Example: $n = 12 = 2^2 \cdot 3^1$. $n_2(G) | 3$, odd so $n_2(G) \in \{1, 3\}$,
$n_3(G) | 4$, so $n_3(G) \in \{1, 4\}$ (2 \neq 1 \times 3)$

Case 1: $n_3(G) < 4$. Then the action of $G$ by conjugation on $Syl_3(G)$
gives a hom $f: G \to S_{Syl_3(G)} \cong S_4$.

What is $\ker(f)$? That consists of those $g \in G$ that normalize all 3-Sylow
subgroups. Let $P_3$ be a 3-Sylow subgroup. Then $|P_3| = 3$, $[G: N_G(P_3)] = 4$

So

$$
\begin{array}{c}
\text{in words: } \frac{|G|}{|N_G(P_3)|} = \frac{12}{3} = 4
\end{array}
$$

Number of conjugates

$$
\begin{array}{c}
\text{so } \frac{1}{|N_G(P_3)|} \left( \begin{array}{c}
\frac{P_3}{3} \\
3
\end{array} \right)_3 = \frac{12}{3} = 4
\end{array}
$$

Now so $\ker(f) = \bigcap \text{Syl}_3(G) = \text{intersection of the 3-Sylow subgroups} = \{e\}$
(intersection of distinct subgroups = $C_3$, is a proper subgroup of
at least one, hence trivial)

Conclusion: $G$ is a subgroup of $S_4$, of order 12.

Want to show $G = A_4$. For this note: if $g \in P_3 \ \neq e$, $g$ has
order 3, so $f(g)$ has order 3, so $f(g)$ is a 3-cycle.
$G$ has 8 elements of order 3: 4 3-sylow subgps each has $3! = 6$ elements of order 3 (these are all distinct because we saw the subgps are disjoint).

$S_4$ has \[
\frac{4 \cdot 3 \cdot 2}{3} = 8
\]
3-cycles: 3-cycle has form $(ijk)$

$S_4$ generates all 3-cycles, hence the subgps they generate which is $A_4$, so $f(G) \geq A_4$ but $|f(G)| = 12 = |A_4|$, so $f(G) = A_4$ and $G \cong A_4$.

**Alternative:** Show that $A_4$ is the only subgroup of $S_4$ of order 12:

Let $H < S_4$ have order 12. By Cauchy, $H$ contains $C$ of order 3, which is a 3-cycle (other cycle structures are $(1)$, $(12)$, $(1234)$, $(1234)$)

But $H$ is normal ($[S_4 : H] = 2$) so $H$ contains all conjugates of $C$.

So $H \triangleleft S_4 \Rightarrow [S_4 : H] = 2$, $H$ is normal.

(PS: if $[G : H] = 2$, $H$ is normal)

(PS: if $[G : H] = p$, $p$ smallest prime $|G|$, then $H$ is normal)

Alternative: $f(G)$ contains $\geq 4$ 3-sylow subgps

Consider $\text{Syl}_3(S_4)$. $\# S_4 = 24 = 8 \cdot 3$ so 3-sylow subgps of $S_4$ are of order 3, and their number $n_3(S_4) \in \{1, 2, 4, 8\}$, $n_3(S_4) = 1 (3)$, so $n_3(S_4) \not\in \{1, 4\}$ but $n_3(S_4) = n_3(f(G))$ so they are equal: $f(G)$ contains all elements of order 3 in $S_4$.

Bottom line: if $|G| = 12$, $n_3(G) = 4$ then $G \cong A_4$. 
Otherwise, \[ n_3(G) = 1, \text{ now the 3-sylow subgp } P \text{ is normal} \]

Let \( Q \) be a 2-sylow subgp, \( \# Q = 4 \) \[ \gcd(\# P, \# Q) = 1 \]

Then \( G = P \times Q \). \[ P \circ Q = \text{?} \] because \( \# P \odot \# Q = \# G \)

Now \( P = C_3 \). Remains: 1) classify \( Q \), (2) classify actions of \( Q \) on \( P \)

1) \( Q \) is isomorphic to one of \( C_4, C_2 \times C_2 \)

2) \[ \text{Case 2a: } Q = C_4 = \{ 1, b, b^2, b^3 \} \]

need action: \( Q \to \text{Aut}(P) = \{ \tau, -\tau \} \)

two homs: either \( \nu(b) = + \), \( \nu \) trivial, get \( C_4 \times C_3 \)

or \( \nu(b) = - \), then \( \nu(b^2) = + \), \( \nu(b^3) = - \).

(this is the map \( \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \) given by "reduction mod 2")

get a non-commutative pgrph: \( C_4 \times C_3 \)

Case 2b: \( Q = C_2 \times C_2 \)

Need to classify \( \text{Hom}(C_2 \times C_2, C_2) \)

either image is trivial, then \( G = (C_2 \times C_2) \times C_3 \)

or: choose \( \varphi, \psi \in \text{Hom}(C_2 \times C_2, C_2) \)

\[ \text{Claim: } Q = \{ 1, a, a_2, a_3, a_4 \} \]

so \( Q \cong C_2 \times C_2 \), where 1st copy of \( C_2 \)

acts on \( C_3 \)

2nd doesn't: \( Q \times C_3 \cong (C_2 \times C_2) \times C_3 \cong C_2 \times (C_2 \times S_3) \cong C_2 \times S_3 \)
Conclusion: Groups of order 12, up to isomorphism are:

$A_4$, $C_3 \times C_4$, $C_3 \times C_4$, $C_2 \times C_2 \times C_3$, $C_2 \times S_3$

$C_{12}$