Math 322: Problem Set 9 (due 19/11/2015)

P1. In class we classified the groups of order 12, finding the isomorphism types $A_{12}$, $C_{12}$, $C_4 \times C_3$, $C_2 \times C_6$, $C_2 \rtimes S_6$. The dihedral group $D_{12}$ is a group of order 12 – where does it fall in this classification?

P2. (Numerology) Let $G$ be a group of order $p^2q$ where $p, q$ are prime.
   (a) Show that, unless $q \equiv 1 \pmod{p}$, $G$ has a unique $p$-Sylow subgroup and isn’t simple.
   (b) Show that, unless $p^2 \equiv 1 \pmod{q}$, $G$ has a unique $q$-Sylow subgroup and isn’t simple.
   (c) Show that if $q \equiv 1 \pmod{p}$ and $p^2 \equiv 1 \pmod{q}$ then $p = 2, q = 3$ and $G$ isn’t simple.

Sylow’s Theorems

Write $P_p$ for a $p$-Sylow subgroup of $G$.

1. Let $G$ be a simple group of order $36 = 2^23^2$.
   RMK The idea of P2 shows that a group of order $p^2q^2$ isn’t simple unless $p^2q^2 = 36$.
   (a) Show that $G$ acts non-trivially on a set of size 4.
   (b) Use the kernel of the action to show $G$ isn’t simple after all.

2. Let $G$ be a group of order $255 = 3 \cdot 5 \cdot 17$.
   (a) Show that $n_{17}(G) = 1$.
   (*b) Show that $P_{17}$ is central in $G$.
   (c) Show that $n_{5}(G) = 1$.
   (d) Show that $P_{5}$ is also central in $G$.
   (e) Show that $G \cong C_3 \times C_5 \times C_{17} \cong C_{255}$.

3. Let $G$ be a group of order 140
   (a) Show that $G \cong H \rtimes C_{35}$ where $H$ is a group of order 4.
   (**b) Classify actions of $C_4$ on $C_{35}$ and determine the isomorphism classes of groups of order 140 with $P_2 \cong C_4$.
   (**c) Classify actions of $V$ on $C_{35}$ and determine the isomorphism classes of groups of order 140 with $P_2 \cong V$.

4. Let $G$ be a finite group, $P < G$ a Sylow subgroup. Show that $N_G(N_G(P)) = N_G(P)$ (hint: let $g \in N_G(N_G(P))$ and consider the subgroup $gPg^{-1}$).

5. Let $G$ be a finite group of order $n$, and for each $p|n$ let $P_p$ be a $p$-Sylow subgroup of $G$.
   (a) Show that $G = \left< \bigcup_{p|n} P_p \right>$.
   (b) Suppose that $G_p$ has a unique $p$-Sylow subgroup for each $p$. Show that $G = \prod_p P_p$ (internal direct product).

(hint for 2b: conjugation gives a homomorphism $G \to \text{Aut}(P_{17})$).
(hint for 2c: let $G$ act by conjugation on $\text{Syl}_5(G)$ and use part b).