Math 322: Problem Set 5 (due 15/10/2015)

Practice problems

P1. Let $N < G$ satisfy for all $g \in G$ that $gNg^{-1} \subset N$. Show that for all $g \in G$, $gNg^{-1} = N$.

P2. Let $N < G$ satisfy for all $g_1, g_2 \in G$ that if $g_1 \equiv_L g_1' (N)$ and $g_2 \equiv_L g_2' (N)$ then $g_1g_2 \equiv_L g_1'g_2' (N)$.

(a) Show that for any $g \in G$, $n \in N$ we have $gng^{-1} \equiv_L e (N)$, and conclude that $gNg^{-1} = N$.

(b) Show that the map $H \rightarrow G/N$ such that $N = \ker (q)$. Conclude that $N$ is normal.

Cosets, normal subgroups and quotients

1. (Normalizers and centralizers) Let $G$ be a group, $X \subset G$ a subset. The centralizer of $X$ (in $G$) is $Z_G(X) = \{ g \in G \mid \forall x \in X : gx = xg \}$ (in particular $Z_G(G)$ is called the centre of $G$).

(a) Show that $N_G(X) < G$.

PRAC Show that $Z_G(X) \subset N_G(X)$.

(b) Show $H < N_G(H)$.

PRAC Let $H < K < G$. Show that $H \triangleleft K$ iff $K \subset N_G(H)$. In particular, $H \triangleleft G$ iff $N_G(H) = G$.

(c) Show that $Z(G)$ is a normal, abelian subgroup of $G$.

PRAC Show that $H \cap Z_G(H) = Z(H)$, in particular that $H \subset Z_G(H)$ iff $H$ is abelian.

2. (Semidirect products) Let $H, K < G$ and consider the map $f : H \times K \rightarrow G$ given by $f (h, k) = hk$. Recall that the image of this map is denoted $HK$.

(a) Show that $f$ is injective iff $H \cap K = \{ e \}$.

SUPP For $x \in HK$ give a bijection $f^{-1}(x) \leftrightarrow H \cap K$, hence a bijection $H \times K \leftrightarrow HK \times H \cap K$.

PRAC Show $H < N_G(K) \iff \forall h \in H : hKh^{-1} = K$. In this case we say “$H$ normalizes $K$”.

(b) Suppose $H$ normalizes $K$. Show that $HK$ is a subgroup of $G$ and that $\langle H \cup K \rangle = HK$.

Show that $K \triangleleft HK$ (hint: you need to show that $HK < N_G(K)$ and already know that $K, K$ separately are contained there).

DEF If $H < N_G(K)$ and $K \cap H = \{ e \}$ we call $HK$ the (internal) semidirect product of $H$ and $K$. We write $HK = H \ltimes K$ (combining the symbols for product and normal subgroup).

(c) Let $HK$ be the semidirect product of $H, K$ and let $q : HK \rightarrow (HK)/K$ be the quotient map.

Directly show that the restriction $q \mid_H : H \rightarrow (HK)/K$ is an isomorphism. (Hint: what is the kernel? what is the image?)

PRAC Let $g, h \in G$. Show that $gh = hg$ if and only if the commutator $[g, h] = ghg^{-1}h^{-1}$ has $[g, h] = e$.

For parts (d), (e) suppose that $H, K$ normalize each other and that $H \cap K = \{ e \}$

(d) Show that $H, K$ commute: $hk = kh$ whenever $h \in H, k \in K$.

(e) Show that the map $f$ is an isomorphism onto its image (it’s a bijection by part (a); you need to show it is a group homomorphism).

DEF In that case we say $HK$ is the (internal) direct product of $H$ and $K$. 

54
3. Let $K < H < G$ be a chain of subgroups. Let $R \subseteq G$ be a system of representatives for $G/H$ and let $S \subseteq H$ be a system of representatives for $H/K$.
   (a) Show that the map $R \times S \to RS$ given by $(r,s) \mapsto rs$ is a bijection.
   (b) Show that $RS = \{rs \mid r \in R, s \in S\}$ is a system of representatives for $G/K$, and conclude that $[G : K] = [G : H][H : K]$.
   RMK See the practice problems file for a numerical proof in the finite case.

4. In a previous problem set we defined the subgroup $G_B$. (The derived subgroup and abelian quotients) Fix a group $G$.
   (a) Show that the map $G \to G/\langle N \rangle$ has this property. Suppose that $G$ contains $N$ together with a homomorphism $\bar{\tau}: G \to G$ such that the property for any $f: G \to H$ with kernel containing $N$ there is a unique $\bar{f}: \bar{G} \to H$ with $f = \bar{f} \circ \bar{\tau}$ (in class we saw that the quotient group $G/N$ has this property). Suppose that $(\bar{G}', \bar{\tau}')$ is another abstract quotient. Show that there is a unique isomorphism $\varphi: \bar{G} \to \bar{G}'$ such that $\bar{\tau}' = \varphi \circ \bar{\tau}$.

5. Let $G$ be a group
   (a) Suppose that $x^2 = e$ for all $x \in G$. Show that $G$ is abelian.
   (**b) Suppose that $G$ has $n$ elements, at least $\frac{3}{4}n$ of which have order 2. Then $G$ is abelian.

6**. Let $G$ be group of order $n$. Show that there is $X \subset G$ of size at most $1 + \log_2 n$ such that $G = \langle X \rangle$.

Bonus problems

Supplementary Problems: Quotients and the abelianization

A. (The universal property of $G/N$) Let $N \triangleleft G$. An “abstract quotient” of a group $G$ is a group $\bar{G}$, together with a homomorphism $\bar{\tau}: G \to \bar{G}$ such that the property for any $f: G \to H$ with kernel containing $N$ there is a unique $\bar{f}: \bar{G} \to H$ with $f = \bar{f} \circ \bar{\tau}$ (in class we saw that the quotient group $G/N$ has this property). Suppose that $(\bar{G}', \bar{\tau}')$ is another abstract quotient. Show that there is a unique isomorphism $\varphi: \bar{G} \to \bar{G}'$ such that $\bar{\tau}' = \varphi \circ \bar{\tau}$.

B. (The derived subgroup and abelian quotients) Fix a group $G$ and recall that notation $[g,h] = ggh^{-1}h^{-1}$.
   (a) Let $f \in \text{Hom}(G,H)$ be a homomorphism. Show that $f([g,h]) = [f(g), f(h)]$ for all $g, h \in G$.
   (b) Deduce from (a) that the set of commutators is invariant under conjugation.
   DEF For $H, K \triangleleft G$ set $[H, K] = \{[h,k] \mid h, k \in G\}$ — note that this is the subgroup generated by those commutators, not just the set of commutators. In particular, we write $G' = [G, G]$ for the derived subgroup (or commutator subgroup) of $G$, the subgroup generated by all the commutators.
   (c) Show that $G'$ is normal in $G$.
   (d) Show that $G^{ab} \overset{\text{def}}{=} G/G'$ is abelian (hint: apply (a) to the quotient map).
   DEF we call $G^{ab}$ the abelianization of $G$.
   (e) Let $N \triangleleft G$. Show that $G/N$ is abelian iff $G' \subseteq N$.
   (f) Let $A$ be an abelian group and let $q: G \to G^{ab}$ be the quotient map. Show that the map $\Phi: \text{Hom}(G^{ab}, A) \to \text{Hom}(G, A)$ given by $\Phi(f) = f \circ q$ is a bijection.

C. Compute the derived subgroup and the abelianization of the groups: $C_n, D_{2n}, S_n, \text{GL}_n(\mathbb{R})$. 55