Math 121: Honours Integral Calculus
Lecture 6

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Last time

- \( f : [a, b] \to \mathbb{R} \) bounded.
- \( P : a = x_0 < x_1 < \cdots < x_n = b \) a partition.
  - Spacings \( \Delta x_i = x_i - x_{i-1} \).
  - Mesh: \( \delta(P) = \max \{ \Delta x_i \}_{i=1}^n \) (longest spacing).
  - \( m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \).
  - Riemann sums \( L(f; P) = \sum_{i=1}^{n} m_i \Delta x_i, \)
    \( U(f; P) = \sum_{i=1}^{n} M_i \Delta x_i. \)

**Definition**

Say \( f \) is integrable on \([a, b]\) and that \( \int_a^b f(x)dx = I \) if \( I \in \mathbb{R} \) is the unique number so that \( L(f; P) \leq I \leq U(f; P) \) for all \( P \).
Examples

- \( f \) constant: \( \int_a^b c \, dx = c(b - a) \).

- Dirichlet’s function \( D(x) = \begin{cases} 1 & \text{x rational} \\ 0 & \text{x irrational} \end{cases} \) is not integrable on any interval \( (m_i = 0, M_i = 1) \).

- What if \( f(x) = \begin{cases} D(x) & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases} \) on \([0, 1]\)?

- What if \( f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases} \)?
The choice of partition

**Theorem**

The following are equivalent:

1. *I* is the unique number between the lower and upper sums;
2. \( I = \lim_{\delta(P) \to 0} L(f; P) = \lim_{\delta(P) \to 0} U(f; P); \)
3. The sums for the uniform partition converge to *I*.

Why non-uniform partitions?

**Example**

\( f(x) = \log x \) on \([a, b]\) \((a > 0)\). The natural partition is \( x_i = a \left( \frac{b}{a} \right)^{i/n} \). After complicated calculation (see notes) get

\[
\int_{a}^{b} \log dx = (b \log b - b) - (a \log a - a).
\]
The interval

**Theorem**

Let $f$ be Riemann integrable on $[a, b]$. Then $f$ is integrable on any sub-interval.

**Theorem**

Let $f$ be integrable on $[a, b]$, $[b, c]$. Then $f$ is integrable on $[a, c]$ and

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$ 

**Proof.**

Concatenate partitions.
The interval

**Definition**

If \( b < a \) set \( \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx \). If \( a = b \) set the integral to zero.

**Example**

\[ f(x) = \begin{cases} 5 & 2 \leq x < 3 \\ -2 & 3 < x \leq 5 \end{cases} \]

Then

\[ \int_2^5 f(x) \, dx = \int_2^3 f(x) \, dx + \int_3^5 f(x) \, dx = 5 \cdot 1 + (-2) \cdot 2 = 1. \]

**Example**

We showed that \( \int_0^b x \, dx = \frac{b^2}{2} \) (right triangle with both sides of length \( b \)). Since \( \int_0^b = \int_0^a + \int_a^b \) it follows that

\[ \int_a^b x \, dx = \int_0^b x \, dx - \int_0^a x \, dx = \frac{b^2}{2} - \frac{a^2}{2}. \]
The function

**Theorem**

Let $f, g$ be Riemann integrable on $[a, b]$. Let $A, B \in \mathbb{R}$. Then $Af +Bg$ is integrable on $[a, b]$ and

$$
\int_a^b (Af(x) + Bg(x)) \, dx = A \int_a^b f(x) \, dx + B \int_a^b g(x) \, dx.
$$

**Example**

$$
\int_a^b (Ax + B) \, dx = A \int_a^b x \, dx + B \int_a^b 1 \, dx = A \left( \frac{b^2}{2} - \frac{a^2}{2} \right) + B(b - a).
$$
The function

Theorem

Let $f$ be continuous on $[a, b]$. Then $f$ is integrable on $[a, b]$.

Proof.

Continuity: $f$ does not fluctuate much on small intervals. But this means that if $\delta(P)$ is small then $U(f; P) - L(f; P)$ is small (at most $b - a$ times the maximal fluctuation). It follows that there is at most one real number between all the lower and upper sums.