Math 121 – Summary of improper integrals
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1 Definitions

- For \( f \) defined for \( x \geq a \) so that \( \int_a^T f(x) \, dx \) makes sense for all \( x \) we set (IF THE LIMIT EXISTS)
  \[
  \int_a^\infty f(x) \, dx = \lim_{T \to \infty} \int_a^T f(x) \, dx
  \]
  - Say the integral “converges” if the limit exists, “diverges” if it doesn’t.
  - The notation on the LHS is shorthand for the value of the limit.
  - \( \int_a^\infty f(x) \, dx \) converges iff \( \int_a^b f(x) \, dx \) converges and in that case \( \int_a^\infty f(x) \, dx = \int_a^b f(x) \, dx + \int_b^\infty f(x) \, dx \) (“area is additive”).
  - Intuition: All that matters is the asymptotic behaviour near infinity.

- For \( f \) defined for \( a < x \leq b \) we set
  \[
  \int_a^b f(x) \, dx = \lim_{T \to \infty} \int_a^T f(x) \, dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x) \, dx.
  \]
  - Again same terminology for “convergence”, “divergence”.
  - Again if \( f \) bounded near \( b \) then value of \( b \) not important – only behaviour near \( a \) is.

- If \( \int_a^b f(x) \, dx \) has several “bad points”, break up into sub-intervals on with one bad endpoint each.
  \[
  \int_0^\infty \frac{e^x}{\sqrt{x^2-2}} \, dx = \int_0^1 \frac{e^x}{\sqrt{x^2-2}} \, dx + \int_1^2 \frac{e^x}{\sqrt{x^2-2}} \, dx + \int_2^3 \frac{e^x}{\sqrt{x^2-2}} \, dx + \int_3^\infty \frac{e^x}{\sqrt{x^2-2}} \, dx.
  \]
- Limit laws apply, so if the integrals involving \( f, g \) on some interval converge so does the integral involving \( \alpha f + \beta g \).

2 \( f \) positive

- Then \( \int_a^T f(x) \, dx \) is increasing when the interval increases. As \( T \to \infty \) it is either bounded (and the limit exists) or unbounded (and the limit is \( \infty \)).
  - Key examples:
    \[
    \int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{T \to \infty} \int_1^T \frac{dx}{\sqrt{x}} = \lim_{T \to \infty} \left[ 2\sqrt{x} \right]_1^T = \lim_{T \to \infty} \left( 2\sqrt{T} - 2 \right) = \infty
    \]
    \[
    \int_1^\infty \frac{dx}{x^2} = \lim_{T \to \infty} \int_1^T \frac{dx}{x^2} = \lim_{T \to \infty} \left[ -\frac{1}{x} \right]_1^T = \lim_{T \to \infty} \left( 1 - \frac{1}{T} \right) = 1.
    \]
3 Absolute convergence

- In general
  \[ \int_1^\infty \frac{dx}{x^p} = \lim_{T \to \infty} \left[ \frac{T^{1-p}}{1-p} - \frac{1}{1-p} \right] = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p \leq 1 \end{cases} \]

- At a finite interval
  \[ \int_0^1 \frac{dx}{x^p} = \lim_{T \to 0} \left[ \frac{1}{1-p} - \frac{T^{1-p}}{1-p} \right] = \begin{cases} \frac{1}{p-1} & p < 1 \\ \infty & p \geq 1 \end{cases} \]

- Comparison
  - For \( f \) positive, all that matters is the upper bound, so: if \( 0 \leq f(x) \leq g(x) \) for all \( x \) then
    - If an improper integral for \( g \) on some interval converges the same holds for \( f \) (smaller area is also finite).
    - If an improper integral for \( f \) on some interval diverges the same holds for \( g \) (larger area is also infinite).
  - Key situation: suppose for \( x \) large \( f, g \) are positive and there are constants \( 0 < A < B \) so that \( A \leq \frac{f(x)}{g(x)} \leq B \) for \( x \) large. Then \( \int_a^\infty f(x) \, dx \) and \( \int_a^\infty g(x) \, dx \) either both converge or both diverge.
  - Examples for deciding convergence:
    1. \( \frac{1}{\sqrt{x^3 - 5}} \sim \infty x^{-3/2} \) (asymptotics as \( x \to \infty \)); since \( \int_{10}^\infty \frac{dx}{\sqrt{x^3-5}} \) converges so does \( \int_{10}^\infty \frac{dx}{\sqrt{x^3}} \).
    2. Decide if \( \int_0^1 \frac{\sqrt{x}}{x} \, dx \) converges. Only bad point is at \( x = 0 \); there we have \( \frac{\sqrt{x}}{x} \sim 0 \frac{1}{\sqrt{x}} \). Since \( \int_0^1 \frac{dx}{\sqrt{x}} \) converges so does \( \int_0^1 \frac{dx}{\sqrt{x^3}} \).
    3. \( \int_{1/2}^1 \frac{dx}{\sin(\pi x)} \). The integrand blows up as \( x \to 1 \). In what way?
      - Method 1: change variables to \( y = 1-x \), so we are looking at \( \int_{1/2}^0 -\frac{dy}{\sin(\pi y)} = \int_0^{1/2} \frac{dy}{\sin(\pi y)} \). Now \( \sin(\pi y) \sim_0 \pi y \) so \( \frac{1}{\sin(\pi y)} \sim_0 \frac{1}{\pi y} \) and the integral diverges.
      - Method 2: (same idea, different presentation) write \( \sin(\pi x) = -\sin(\pi x - \pi) = -\sin(\pi(1-x)) \).
        As \( x \to 1 \), \( 1-x \) is small so \( \sin(\pi(1-x)) \sim_1 \pi(1-x) \). It follows that
        \[ \frac{1}{\sin(\pi x)} \sim \frac{1}{x-1} = \frac{1}{1-x} \] 

        Now \( \int_{1/2}^1 \frac{dx}{\sin(\pi x)} \) diverges since the integrand blows at rate \( \frac{1}{\text{distance to bad point}} \).

3 Absolute convergence

- Suppose \( \int_a^\infty |f(x)| \, dx \) converges. Then \( g(x) = f(x) + |f(x)| \) satisfies \( 0 \leq g(x) \leq 2|f(x)| \) so \( \int_a^\infty (f(x) + |f(x)|) \, dx \) converges. Also, \( \int_a^\infty (-|f(x)|) \, dx \) converges. Adding we see that \( \int_a^\infty f(x) \, dx \) converges.
  - If \( \int_a^\infty |f(x)| \, dx \) converges we say \( \int_a^\infty f(x) \, dx \) converges absolutely.
  - If \( \int_a^\infty f(x) \, dx \) converges but \( \int_a^\infty |f(x)| \, dx = \infty \) we say \( \int_a^\infty f(x) \, dx \) converges conditionally.

- Key examples:
  - \( \int_0^\infty \frac{\cos x}{x^2} \, dx \) converges absolutely since \( \left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2} \).
  - \( \int_0^\infty \frac{\cos x}{x} \, dx \) converges conditionally.