Math 121: Problem set 3 (due 27/1/12)

Practice problems (not for submission!)

Section 6.1-6.3: all problems, especially those marked "challenging". Ignore problems for computer-assisted exploration.

Estimation

1. Simplifying integrals.
   (a) Show that \( \int_{1}^{\frac{3}{2}} \sqrt{x^2-1} \, dx \leq \frac{3}{2} \).
   \textbf{Hint:} \( \sqrt{x^2-1} \leq \sqrt{x^2} \).
   (b) Evaluate \( \lim_{T \to \infty} \frac{1}{T} \int_{T}^{T+1} \sqrt{x^2-1} \, dx \).

2. Estimating \( \log(2) \) and \( \pi \).
   (a) Show that \( \int_{0}^{1} \frac{1}{1+t^4} \, dt = \frac{\pi}{4} \).
   \textbf{Hint:} What is \( \frac{d}{dx}(\arctan x) \)?
   (b) Show that \( 1-t^2 \leq \frac{1}{1+t^2} \leq 1-t^2+t^4 \) and conclude that \( \frac{8}{5} \leq \pi \leq \frac{52}{15} \).
   (c) Show that \( \int_{0}^{1} (x-x^2)^4 \, dx = 22 \frac{7}{3} - \pi \).
   \textbf{SUPP} Show that \( \int_{0}^{1} (x-x^2)^4 \, dx = 22 \frac{7}{3} - \pi \).
   \textbf{SUPP} Show that \( \frac{1979}{630} \leq \pi \leq \frac{22}{7} \).

3. Let \( f, g \) be continuous on the interval \([a, b] \).
   \textbf{SUPP} Show that \( \int_{a}^{b} (f(x))^2 \, dx = 0 \) implies that \( f(x) = 0 \) for all \( x \).
   \textbf{Hint:} Assuming \( f \) is non-zero somewhere show that it is non-zero on an entire subinterval, and construct a non-zero lower Riemann sum for \( f^2 \) on \([a, b] \).
   (b) Assuming that \( f \) is not identically zero, find the point \( t_0 \) where the function \( G(t) \) below achieves its global minimum.
   \[ G(t) = \int_{a}^{b} (tf(x) + g(x))^2 \, dx. \]
   (c) Show that \( G(t_0) \geq 0 \) and deduce the \textit{Cauchy-Schwartz inequality}
   \[ \left( \int_{a}^{b} f(x)g(x) \, dx \right)^2 \leq \left[ \int_{a}^{b} (f(x))^2 \, dx \right] \left[ \int_{a}^{b} (g(x))^2 \, dx \right]. \]
   – What about the case where \( f \) is identically zero?

Techniques of integration

4. Let \( I_n = \int x^{2n} \cos x \, dx, J_n = \int \sin^n x \, dx. \)
   (a) Obtain a reduction formula for \( I_n \).
   \textbf{Hint:} Textbook page 335.
   (b) Obtain a reduction formula for \( J_n \).
   \textbf{Hint:} Textbook page 336.
   (c) Use your formula to calculate \( \int_{-\pi/2}^{\pi/2} x^{2n} \cos x \, dx \) for \( n = 3 \).
Supplementary problem – the substitution rule for discontinuous functions

A. Let \([a, b]\) be an interval, and let \(g : [a, b] \to \mathbb{R}\) be continuously differentiable with positive derivative. Let \(c = g(a)\) and \(d = g(b)\).

(a) Show that \(g\) is surjective (“onto”) the interval \([c, d]\): that for \(u \in [c, d]\) there is \(x \in [a, b]\) with \(g(x) = u\).

(b) Show that \(g\) is injective (“one-to-one”): the \(x\) in part (a) is unique. We write \(g^{-1}(u)\) for the unique \(x\) solving \(g(x) = u\).

(c) Let \(P : a = x_0 < \cdots < x_n = b\) be a partition of \([a, b]\). Show that setting \(u_i = g(x_i)\) gives a partition of \([c, d]\), to be denoted \(g(P)\).

(**d) Let \(f\) be bounded on \([c, d]\) and let \(\varepsilon > 0\). Show that if the mesh \(\delta(P)\) is small enough then \(|U(f; g(P)) - U((f \circ g)g'; P)| \leq \varepsilon\) and \(|L(f; g(P)) - L((f \circ g)g'; P)| \leq \varepsilon\).

(e) Suppose that \((f \circ g)g'\) is integrable on \([a, b]\), or that \(f\) is integrable on \([c, d]\). Show that the other function is integrable as well and that \(\int_a^b f(g(x))g'(x)\,dx = \int_c^d f(u)\,du\).

RMK Note that \(f\) was not assumed continuous.