1. Consider the function \( f: [0, 1] \to [0, 1] \) given by \( f(x) = 3x^2 - 2x^3 \). The cell structure on \([0, 1]\) (consisting of two 0-cells and one 1-cell) induces a cell structure on \([0, 1]^n\); call this the obvious cell structure on the hypercube.

(i) Check that \( f \) is a Morse function on \([0, 1]\).

(ii) Define \( F(x_1, x_2, x_3) = f(x_1) + f(x_2) + f(x_3) \) for \( x_i \in [0, 1] \) and show that \( F \) is Morse.

(iii) Show that \( -\nabla F \) induces the obvious cell structure on \([0, 1]^3\).

(v) Give the definition of a 2-manifold with corners, and give an explicit description of the set of flow lines of \( -\nabla F \) as a 2-manifold with corners.

2. Let \( f: M \to [0, 3] \) be a self-indexing Morse function on a space \( M \), which has a unique maximum value and a unique minimum value. The picture below describes \( f^{-1}(\frac{3}{2}) \), which is a genus 2 surface obtained from \( S^2 \) (the page, compactified) after identifying each pair of holes at the same height (without any twisting):

You should assume that the two red curves consist of points flowing to index 1 critical points (along \( -\nabla f \)) and the two blue curves consist of points flowing to the index 2 critical points (along \( \nabla f \)).

(i) Using the description above, calculate the cellular homology groups \( H_i(M; \mathbb{Z}) \).

(ii) Calculate \( \pi_1(M) \), noting that each blue curve describes a 2-cell attaching map.

(iii) Find a surjection from \( \pi_1(M) \) to \( A_5 \).
3. Determine $H_1(\hat{X}; k)$ and $H_0(\hat{X}; k)$ as $k[t, t^{-1}]$-modules, where $X$ is the Klein bottle with infinite cyclic cover $\hat{X}$ and $k$ is any field.

4. Let $X$ be the complement of a knot $K: S^1 \hookrightarrow S^3$, and let $\hat{X}$ be the infinite cyclic cover determined by the Hurewicz map. Recall that if $p: \hat{X} \to X$ is the associated covering map, then

$$p_*: \pi_1 \hat{X} \to \pi_1 X$$

is a homeomorphism with image $C$ isomorphic to the commutator subgroup $[\pi_1 X, \pi_1 X]$ in $\pi_1 X$. The key observation for this problem is that $p_*$ induces an isomorphism

$$\bar{p}_*: H_1(\hat{X}; \mathbb{Z}) \to \frac{C}{[C, C]}$$

Write $\Lambda$ for the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$.

(i) Suppose $c \in C$ and let $x \in \pi_1 X$ be an element sent to a generator under abelianization. Define $t[c] = [xcx^{-1}]$ where $[\cdot]$ denotes the coset in $C/[C, C]$. Show that $t$ is a well-defined automorphism of $C/[C, C]$.

(ii) Let $t$ be the generator of $\Lambda$ acting on $H_1(\hat{X}; \mathbb{Z})$. Show that, up to appropriate choices of generators, $\bar{p}_* \circ t = t \circ \bar{p}_*$. (This promotes $\bar{p}_*$ to an isomorphism of $\Lambda$-modules.)

(iii) Show that $C$ is generated by all words of the form $x^k g_i^{\pm 1} x^{-k}$ ($x$ as above) if the $g_i$ generate $\pi_1 X$, so that the substitution

$$\pm t^{k\gamma_i} \leftrightarrow x^k g_i^{\pm 1} x^{-k}$$

gives rise to a $\Lambda$-module presentation for $C/[C, C]$.

(iv) Let $K$ be the trefoil (as see in class) so that

$$\pi_1 X \cong \langle x, y | xyx = yxy \rangle$$

Setting $a = yx^{-1}$, and following the strategy suggested above, determine $\Lambda$-module structure on $C/[C, C]$.

5. As seen in class, any $K: S^1 \to S^3$ bounds an orientable surface $F$. Let $X$ be the complement of the trefoil knot, and consider the 2-fold cover $Y \to X$ obtained by sending $\pi_1 X$ onto $\mathbb{Z}/2\mathbb{Z}$. Using the Mayer-Vietoris sequence, calculate $H_1(Y; \mathbb{Z})$. 
