1. Universal properties.

(i) Recall the definition of a product in an arbitrary category, and check that the product of spaces is an example of a product in \( \text{Top} \). Show that the universal property uniquely determines the product topology.

(ii) Starting with the definition of the quotient topology via universal properties, give a definition for the quotient topology in terms of open sets.

2. Fix spaces \( X \) and \( Y \).

(i) Show that \( f \simeq g \) is an equivalence relation, for maps \( f, g \colon X \to Y \) admitting a homotopy between them.

(ii) If \( f_0 \) is homotopic to \( f_1 \) and \( g_0 \) is homotopic to \( g_1 \), where the range of \( f \) agrees with the domain of \( g \), show that \( f_0 \circ g_0 \) is homotopic to \( f_1 \circ g_1 \).

(iii) Prove that concatenation of paths \( I \to X \) forms a well-defined product on homotopy classes of paths.

3. Let \( \gamma, \varphi, \psi \) be loops based at a point \( x_0 \) in a space \( X \). Find an explicit homotopy \( H \colon I \times I \to X \) with the properties that \( H(s, 0) = (\gamma * \varphi * \psi)(s) \), \( H(s, 1) = (\gamma * (\varphi * \psi))(s) \) and \( H(s, \frac{1}{2}) = (\gamma * \varphi * \psi)(s) \). Here

\[
(\gamma * \varphi * \psi)(s) = \begin{cases} 
\gamma(3s) & s \in [0, \frac{1}{3}] \\
\varphi(3s) & s \in \left(\frac{1}{3}, \frac{2}{3}\right) \\
\psi(3s) & s \in \left(\frac{2}{3}, 1\right]
\end{cases}
\]

as introduced in class.

4. Consider loops \( \gamma \) and \( \varphi \) so that \([\gamma]\) and \([\varphi]\) are elements of the group \( \pi_1(X, x_0) \). Prove that \( [\gamma][\varphi] = [\text{id}] \) if and only if \( \varphi \simeq \bar{\gamma} \), where \( \bar{\gamma}(s) = \gamma(1 - s) \).

5. The groups \( \pi_1(X, x_0) \) and \( \pi_1(X, x_1) \) are isomorphic for any \( x_0 \neq x_1 \) in a path connected space \( X \). Prove this.