LECTURE 1: PROJECTIONS.

Consider points \((x, y, z) \in \mathbb{R}^3\) subject to:

\[
[A] \quad x^2 + y^2 = 1 \quad \text{and} \quad z = 0
\]

These are points on a circle in the \(xy\)-plane; they form a closed loop in \(\mathbb{R}^3\).

DEF A "KNOT" IS A CLOSED CURVE IN \(\mathbb{R}^3\) THAT DOES NOT INTERSECT ITSELF.

NOTE PERIODICITY IN THIS DEFINITION.

This is easier to see, in our example, if we parameterize the circle:

\[
t \mapsto (\cos t, \sin t, 0)
\]

\[
x(t) = \cos t \quad y(t) = \sin t
\]

\[
z(t) = 0
\]

And

\[
x^2 + y^2 = \cos^2 t + \sin^2 t = 1
\]

so that \([A]\) is satisfied.

In picture:

\[
\begin{align*}
x(t) & \quad \rightarrow \quad t \\
y(t) & \quad \rightarrow \quad t
\end{align*}
\]
Now consider the surface

\[ x = (2 + \cos \theta) \cos \psi \]
\[ y = (2 + \cos \theta) \sin \psi \]
\[ z = \sin \theta \quad \text{where} \quad C \leq \theta, \quad \psi \leq 2\pi \]

You've encountered this as a "Surface of Revolution"

As \( \theta \) varies,

\[ (x, y, z) = (3 \cos \psi, 3 \sin \psi, 0) \]

\( z \) is the "axis of revolution".

(Picture: for \( \theta = 0 \), \( z = \sin(0) = 0 \)).

This surface is a "torus"; call it \( T \).

Note that \( T \subset \mathbb{R}^3 \) (a subset).

Note: for \( \theta = \pi \), \( \psi = t \) we get:

\[ (x, y, z) = (\cos t, \sin t, 0) \]

So our knot is a subset of \( T \).

(\( \text{in red} \))

Other subsets: \( \theta = \frac{\pi}{2}, \psi = t \)
(\( \text{in blue} \)) \( \theta = 0, \psi = t \)
Now let's project to the $xy$-plane.

(View from top; set $z = 0$)

We'd like to view these as 3 different projections of the same knot.

DEF Two knots are equivalent if one may be continuously deformed to the other, in particular, at any stage of this deformation you should still have a knot.

Some projections are bad:

Exp projecting $(\cos t, \sin t, 0)$ to $xy$-plane

Lose too much info

Exp for our example, only projection planes containing $z$-axis cause this issue... so, most proj's OK!
Now consider a larger class of knots:

\[ \theta \rightarrow pt, \ \psi \rightarrow q^t, \ \ p,q \in \mathbb{Z} \]

**EX:** Check that we've been looking at \( p=0, q=1 \):

\[
\left( \frac{(2+\cos(pt)) \cos(q^t)}{x}, \frac{(2+\cos(pt)) \sin(q^t)}{y}, \frac{\sin(pt)}{z} \right)
\]

For each \( p,q \) relatively prime, this is a knot.

**EX:** \( p=2, q=1 \):

\[
\begin{align*}
  x &= (2+\cos(2t)) \cos(t) \\
  y &= (2+\cos(2t)) \sin(t) \\
  z &= \sin(t)
\end{align*}
\]

![Graphs and diagrams showing the behavior of the functions and knots.]

Still the same knot... but new issue in this projection.

But unlike the \( (\cos t, \sin t, 0) \) example, which had a bad projection, this one can be fixed if we keep track...