TRIGONOMETRY PRIMER

This document will introduce (or reintroduce) the concept of trigonometric functions. These functions (and their derivatives) are related to properties of the circle and have many interesting properties, such as periodicity (said when the function repeats itself). We will introduce a new way to measure angles, the radian. We will then properly define the main two functions (sine and cosine) and their properties, all demonstrated with graphs. We will then discuss other related functions. Finally, we will find their derivatives as well as interesting properties they share.

1. THE UNIT CIRCLE

Say we are walking along a circle in the counterclockwise direction. After a short while, we take a break and look back where we started. What is our position now related to that starting point? Suppose we started on the easternmost point and stopped a short while afterwards, we almost walked straight to the north, but not quite, as we veer ever so slightly to the west. What we know is that if we trace lines connecting both positions to the center of the circle, an angle is formed. For this circle, each angle represents a different distance travelled from the starting point and, more importantly, a point on the plane (the place where we took a break, for example). It would be very useful to have functions that connect the angle travelled to the coordinates of the point. This is the role of the trigonometric functions sine and cosine. Trigonometric functions are very important functions when we work with rotations or circles. Going back to our example, we can build a right triangle with our endpoint, the origin, and the point on the eastern direction (x-axis) with same x coordinate as our endpoint. The sides of this triangle have length equal to the two coordinates of the endpoint and the radius of the circle. However, the position of our endpoint depends on that length of the radius, as if we walk the same angle along a very big circle, we will go much further than if we go along a tiny circle. For that reason, the sine and cosine functions will give us the position along a circle of radius 1 (unit circle) and the sides or our triangle will be 1, cos(angle) and sin(angle).

Consider a circle of radius 1 centered at the origin and consider a segment of length 1 connecting the origin with the edge of the circle. The segment connecting any point on this circle with the origin and the positive x-axis do an angle, that we will call \( \theta \) (measured counter-clockwise). The sine and cosine functions will be defined respectively as the x and y coordinates of the end of the segment that touches the circle. Notice that for both function, at any point, if we do a full rotation (2\( \pi \) angle), we get to the same point.

**Definition 1.1.** The sine function (cosine function) is the function that takes an angle \( \phi \) and gives the y axis (x axis) coordinate of the unit circle centered at the origin at an angle \( \phi \) from the positive x-axis in a counter-clockwise direction.

**Remark 1.2.** If we wish to know the position along a circle of different radius, we can use the similarity of triangles and multiply each side by the radius as the ratios of length must be preserved.

Note that when invoking angles I never specify degrees. That is because there are many ways to measure angles, as with distances (meters vs yards). However, when working with trigonometric functions, the radian is the better way to measure angles.

1.1. **Introduction: Why Using Radians?** Most of you should know that cars have a speedometer that display the speed of the car at any given moment. How does it measure that? It cannot draw a table and compute the limit of the rate of change of distance travelled like you. However, it does know how fast the wheels are rotating and from that, can calculate the instant speed from the circumference of the tires. Up until now, me measure the angle of rotation of an object with the angle it makes from a previous position and
we usually measure this angle in degrees, knowing that a full rotation corresponds to 360 degrees. However, it is not simple to find values like the distance travelled from rotation of tires with degrees. This is where radians are useful. Using radians, if a radius 1 tire does a rotation of 1 radian, it travelled a distance of 1 (its radius length). This is more convenient in this situation as a rotation would correspond to a circumference of the wheel in distance, which is $2\pi \text{radius}$, thus, when using a radius 1 tire, a full rotation corresponds to $2\pi$ radians.

**Definition 1.3.** A *radian* is the angle for which the arc traced by a segment rotated at one end is the same length as that segment.

1.2. **Properties of Radians.**

(1) Radians have no unit! (no symbol)

(2) A full rotation is $2\pi$ radians.
(3) The quick conversion from degrees to radians is to multiply by \( \frac{2\pi}{360^\circ} \).

**Example.** Compute 45° in radians and \( \pi/6 \) in degrees.

Solution: A full rotation corresponds to 2\( \pi \) radians or equivalently 360°, therefore the ratio of radians to degrees is \( \frac{2\pi}{360^\circ} \) and 45° \( \times \frac{2\pi}{360^\circ} = \frac{45}{360} \times 2\pi = \frac{2\pi}{8} = \pi/4 \). Similarly, \( \pi/6 \) in degrees is \( \pi/6 \times \frac{360^\circ}{2\pi} = \frac{360^\circ}{12} = 30^\circ \).

1.3. **Properties of Sine and Cosine Functions.** All these properties also hold for the function \( \cos(\phi) \)

1. The domain of the function is all the real numbers.
2. The function is periodic with period 2\( \pi \) (\( \sin(\phi + 2\pi) = \sin(\phi) \)).
3. The function is bounded between -1 and 1 (\( -1 \leq \sin(\phi) \leq 1 \) or \( \text{Im}(\sin(\phi)) = [-1, 1] \))
4. The following property is important: \( \sin^2(\phi) + \cos^2(\phi) = 1 \) (Pythagorean theorem).
5. The angle is always measured in the counter-clockwise direction. Clockwise direction angle are always either measured the long way around or are considered as negative angles.
All these properties hold for all \( \phi \) real.

Note that the sine and cosine functions also represent ratios between two edges of right triangles. In fact, one could see such a triangle in the trigonometric (unit) circle diagram. In this case, the hypotenuse will always be of length 1. Now consider any right triangle. You can imagine this angle being at the origin, but the sine and cosine will not give you the coordinate of the other two vertices. However, because the triangle is similar with the triangle with hypotenuse 1 of the same angle, the ratio between the different edges’ length is the same. Therefore, we can define the sine as the length ratio between the opposite side to the angle and the hypotenuse and the cosine as the length ratio of the adjacent side to the angle and the hypotenuse. This way of defining the sine and cosine are very useful in computations. How about the third length ratio, opposite by adjacent? This defines another trigonometric function, the tangent, and it can be easily determined from sine and cosine.

\[
\tan(\phi) = \frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{\text{hyp}} \times \frac{\text{hyp}}{\text{adj}} = \frac{\sin(\phi)}{\cos(\phi)}
\]

This last step could have easily been found from the unit circle diagram by doing the ratio of the y and x coordinates of the point on the circle.

Remark 1.4. (1) It is important to remember that tangent is always sine over cosine, not the other way around.

(2) Note that the tangent is not defined at \( \pi/2 \) (why?) because that would imply a division by zero. (Any other point where it is undefined?).

(3) The tangent is the slope of the hypotenuse.

Example. Compute \( \sin(\pi/2), \cos(\pi/3), \sin(45^\circ) \) exactly (fractions and roots).

Solution:

(1) For \( \sin(\pi/2) \), notice that the angle \( \pi/2 = 2\pi/4 \) corresponds to one quarter of rotation. Thus, on our unit circle, a quarter rotation from the x-axis in the CCW direction brings us to the \((0,1)\) point. The sine function being the y coordinate of that point, we have that \( \sin(\pi/2) = 1 \) (What would be \( \cos(\pi/2) \)?)

(2) For \( \cos(\pi/3) \), the angle is \( \pi/3 = 2\pi/6 \) and corresponds to one sixth of a rotation (or 60 degrees). That would make the third angle 30 degrees because the sum of internal angles of a triangle is always 180 degrees \((180 = 90 + 60 + 30)\). What we have here is a 90-60-30 triangle and is one of the few triangles with known ratios. These are 1, 2 and \( \sqrt{3} \) [Picture]. Then the cosine will be the ratio between adjacent side and hypotenuse. \( \cos(\pi/3) = \text{adj}/\text{hyp} = 1/2 \).

(3) Consider the right triangle with a 45 degree angle. From the sum of internal angles the third angle will also be 45 degrees. therefore we have an isosceles triangle. If the two sides of equal length were length 1. By Pythagorean theorem the hypotenuse would be \( \text{hyp} = \sqrt{1^2 + 1^2} = \sqrt{2} \). Therefore, sine of 45 degrees (or even better of \( \pi/4 \)) is \( 1/\sqrt{2} = \sqrt{2}/2 \).

1.4. Some Identities of Sine and Cosine.

(1) \( \cos(-x) = \cos(x) \). The cosine function has symmetry with negative angles.

(2) \( \sin(-x) = -\sin(x) \). The sine function is anti-symmetric with negative angles.

(3) \( \sin(\pi - x) = \sin(x) \). Can be shown on the circle.

(4) \( \cos(\pi - x) = -\cos(x) \). Idem

(5) \( \sin(x + 2\pi) = \sin(x) \). Same with cos, tangent.

(6) \( \sin(x + \pi) = -\sin(x) \). Same with cos.

(7) \( \sin^2(\phi) + \cos^2(\phi) = 1 \)
2. Values of Sine and Cosine

Now that we have a rigourous definition of the sine, cosine and other trigonometric functions, let’s draw the graphs of \( f(x) = \sin(x) \), \( \cos(x) \) and \( \tan(x) \). To do this, we will make a table of points that will help us with the sketch.

First, however, we are able to compute the values of sine and cosine using two well known triangles. The first is the right isosceles triangle. This triangle has two \( \frac{\pi}{4} \) angles and its sides’ length are 1, 1 and \( \sqrt{2} \) respectively (see figure 3). Using the ratios of lengths, we find that:

\[
\cos \left( \frac{\pi}{4} \right) = \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}.
\]

![Figure 3. Isosceles right triangle.](image)

The other triangle we can use is the equilateral triangle (!). Even though it is not a right triangle, we can cut it in half as in figure 4 to get a right triangle. This triangle will have a \( \frac{\pi}{3} \) and a \( \frac{\pi}{6} \) angle. We can then compute sine and cosine for these two angles:

\[
\sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}, \quad \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}, \\
\sin \left( \frac{\pi}{6} \right) = \frac{1}{2}, \quad \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}.
\]

![Figure 4. Equilateral triangle cut in half to form a right triangle.](image)

We can now go on and fill our table:
Note that some of these points were already found earlier and most can be found by the properties of the sine and cosine function. The rest can be found using our trigonometric circle. We could extend our table to higher (or lower) $x$ values, for example $x = 2\pi$, but this corresponds to the same value for the three functions as $x = 0$. The same applies for any $x$ value outside the interval $[0, 2\pi)$, therefore we only need to use values inside that interval to get the shape of the function and the rest will repeat itself for every $2\pi$ cycle.

The last figures demonstrate the identities found earlier.
Figure 5. Demonstration of the value of sine and cosine at $\pi - \theta$. 
Figure 6. Demonstration of the value of sine and cosine at $-\theta$. 
Figure 7. Demonstration of the value of sine and cosine at $\theta + \pi$. 